News about the Binomial Formula

Take a look at

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X X X Y
X X X Y
X X X Y
Y Y Y Y
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that is

$$4 \cdot 4 = 3 \cdot 3 + 4 + 3$$

You can generalize

$$(a+1)^2 = a^2 + (a + 1) + a$$

You get this as you start from the inner square and go along the edges.

With the known formula

$$(a + 1)^3 = a^3 + 3 \cdot a^2 + 3 \cdot a + 1$$

and the approach

$$\begin{aligned}(a + 1)^3 &= a^3 + (a + 1)^2 + x + a^2 = \\ &= a^3 + a^2 + 2 \cdot a + 1 + a^2 + x = \\ &= a^3 + 2 \cdot a^2 + 2 \cdot a + 1 + x\end{aligned}$$

you get the following equation

$$x = a^2 + a = a \cdot (a + 1)$$

So you come to the conjecture

$$\forall n \in \mathbb{N}_+ \quad (a + 1)^n = a^n + \sum_{i=0}^{n-1} a^i (a + 1)^{n-1-i}$$

Pre.: Let $\mathfrak{R}$ be a ring with identity unit 1.

Ass.:

$$\forall n \in \mathbb{N}_+ \quad \underbrace{\forall a \in \mathfrak{R} \quad (a + 1)^n = a^n + \sum_{i=0}^{n-1} a^i (a + 1)^{n-1-i}}_{\Leftrightarrow: \; A(n)}$$

Rem.1 Because $\mathfrak{R}$ is a ring with identity unit 1, the following is true:

$$\forall n \in \mathbb{N}_+ \quad \forall a \in \mathfrak{R} \quad (a + 1) \cdot a^n = a^{n+1} + a^n = a^n \cdot (a + 1)$$

Rem.2 With the same premise you get

$$\forall n \in \mathbb{N}_+ \quad \forall a \in \mathfrak{R} \quad (a - 1)^n = a^n - \sum_{i=0}^{n-1} a^i (a - 1)^{n-1-i}$$

Proof.: With the method of complete induction:
The basis of the induction $A(1)$ is trivial.
Let $k \in \mathbb{N}_+$ and assume $A(k)$. Then the following has to be shown:

$$\forall a \in \Re \quad (a+1)^{k+1} = a^{k+1} + \sum_{i=0}^{k} a^i (a+1)^{k-i}$$

Proof of this:

Let $a \in \Re$. With $A(k)$ you get the following:

$$\begin{aligned}
(a+1)^{k+1} &= (a+1) \cdot (a+1)^k = \\
&= (a+1) \cdot \left(a^k + \sum_{i=0}^{k-1} a^i \cdot (a+1)^{k-1-i} \right) = \\
&= (a+1) \cdot a^k + (a+1) \cdot \sum_{i=0}^{k-1} a^i \cdot (a+1)^{k-1-i} = \\
&= a^{k+1} + a^k + \sum_{i=0}^{k-1} a^i \cdot (a+1)^{k-i} = \\
&= a^{k+1} + a^k \cdot (a+1)^0 + \sum_{i=0}^{k-1} a^i \cdot (a+1)^{k-i} = \\
&= a^{k+1} + \sum_{i=0}^{k} a^i \cdot (a+1)^{k-i}
\end{aligned}$$

Thus $A(k+1)$ is proved.