A Striking Proof

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The Problem:

I found the striking proof in "Karl Heinz Mayer, Algebraische Topologie, Birkhäuser Verlag (Basel), ISBN 3-7643-2229-2, 6.7 (page 59)".

It contains a double assumption (in series, not parallel). I feel sick about it. I don't know how to finish the proof. Perhaps it is not allowed at all to do it this way.

It is no problem to prove the thesis of the theorem. It is simple.

The problem is the method of double assumption.

The Simple Proof:

Theo.

Pre. Let X be a topological space.
Let A a connected subset of X.

Ass. $\forall B \subseteq X \quad (A \subseteq B \subseteq \overline{A} \quad \Rightarrow \quad (B \text{ is connected}))$

Proof: Let $B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$.
Let $U_1, U_2 \in \text{Top}(X)$ with

$$B = (U_1 \cap B) \cup (U_2 \cap B)$$
$$\varnothing = (U_1 \cap B) \cap (U_2 \cap B)$$

Because $A \subseteq B$, the following is true:

$$A = (U_1 \cap A) \cup (U_2 \cap A)$$
$$\varnothing = (U_1 \cap A) \cap (U_2 \cap A)$$

Because A is connected, there exists $i \in \{1, 2\}$ with

$$U_i \cap A = \varnothing$$

Because $A \subseteq X$, it follows:

$$A \subseteq X \setminus U_i \qquad (*)$$

Because $U_i \in \text{Top}(X)$, it is:

$$X \setminus U_i \text{ is closed} \qquad (**)$$

With the definition of $\overline{A}$, (*) and (**) we have:

$$\overline{A} \subseteq X \setminus U_i$$

especially with $B \subseteq \overline{A}$

$$B \subseteq X \setminus U_i$$

and finally

$$U_i \cap B = \varnothing$$

The Striking Proof:

Theo.

Pre. Let X be a topological space.
Let A a connected subset of X. (1)

Ass. $\forall B \subseteq X \quad \left(A \subseteq B \subseteq \overline{A} \quad \Rightarrow \quad (B \text{ is connected})\right)$

Proof: Let $B \subseteq X$ with $A \subseteq B \subseteq \overline{A}$.

THE FIRST ASSUMPTION: B is not connected

Then there exists $U_1, U_2 \in \mathrm{Top}(X)$ with

$$
\begin{aligned}
&B = (U_1 \cap B) \cup (U_2 \cap B) \\
&\varnothing = (U_1 \cap B) \cap (U_2 \cap B) \qquad (2) \\
&((U_1 \cap B) \neq \varnothing) \quad \wedge \quad ((U_2 \cap B) \neq \varnothing)
\end{aligned}
$$

Because $A \subseteq B$, the following is true:

$$
\begin{aligned}
&A = (U_1 \cap A) \cup (U_2 \cap A) \qquad (3) \\
&\varnothing = (U_1 \cap A) \cap (U_2 \cap A)
\end{aligned}
$$

Because $B \subseteq \overline{A}$, it follows:

$$
(U_1 \cap A \neq \varnothing) \quad \wedge \quad (U_2 \cap A \neq \varnothing) \qquad (4)
$$

THIS IS A CONTRADICTION TO (1)!

NOW THE PROOF OF (4):

THE SECOND ASSUMPTION: $\left(U_1 \cap A = \emptyset\right) \quad \vee \quad \left(U_2 \cap A = \emptyset\right)$

Then there exists $i \in \{1, 2\}$ with

$$U_i \cap A = \emptyset$$

Because $A \subseteq X$, it follows:

$$A \subseteq X \setminus U_i \tag{5}$$

Because $U_i \in \mathrm{Top}(X)$, it is:

$$X \setminus U_i \text{ is closed} \tag{6}$$

With the definition of $\overline{A}$, (5) and (6) we have:

$$\overline{A} \subseteq X \setminus U_i$$

especially with $B \subseteq \overline{A}$

$$B \subseteq X \setminus U_i$$

and finally

$$U_i \cap B = \emptyset$$

THIS IS A CONTRADICTION TO (2)!
ONE (OR BOTH?) OF THE TWO ASSUMPTIONS FAILED, BUT I CAN'T DECIDE WHICH! I BREAK HERE (I DO NOT KNOW HOW TO GO ON)!