A
Modification
Of
Euler's
Polygonal Method

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Euler's Polygonal Method:

Let $t_0 \in \mathbb{R}$.
Let $b \in \mathbb{R}$ with $t_0 < b$.
Let M be a open subset of $\mathbb{R}$ with $M \neq \varnothing$.
Let $f : M \times \left[t_0, b\right] \to \mathbb{R}$ be a mapping.
Let $\eta \in M$.
Let $h \in \mathbb{R}_+$.

Take a look at the "Initial Value Problem":

$$y' = f(y, t)$$
$$y\left(t_0\right) = \eta$$

With the following recursion you get an approximation for this IVP

$$y_{i+1} := y_i + hf\left(y_i, t_i\right)$$
$$t_{i+1} := t_i + h$$

This recursion is called "Euler's Polygonal Method". It is well known and can be found in "Stoer, Bulirsch: Numerische Mathematik 2, Springer-Lehrbuch".

For the number $n \in \mathbb{N}_0$ of steps in this method we have:

$$n \leq \mathrm{FLOOR}\left(\frac{\left(b - t_0\right)}{h}\right)$$

Modification:

Take a look at the following example $f : \mathbb{R} \times [t_0, b] \to \mathbb{R}$ with $b \to 1,\ b < 1$:

$$\forall t \in [t_0, b] \quad \forall y \in \mathbb{R} \quad f(y, t) := \sin\left(\frac{1}{1-t}\right)$$

In this example you must adapt $h \in \mathbb{R}_+$ for f to get reasonable results. My suggestion is:

$$t_{i+1} := t_i + \frac{h}{\sqrt{1 + \left(f\left(y_i, t_i\right)\right)^2}}$$

$$y_{i+1} := y_i + \left(t_{i+1} - t_i\right) f\left(y_i, t_i\right)$$

The number of steps in this method depends on b, t_0, h and f. Cave: Check for endless loop!

Explanation:

The suggestion is the result of trying to run through the recursion with arc length = 1. For any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and any continuously differentiable mapping $g : [\alpha, \beta] \to \mathbb{R}$ you have:

$$\forall t_1, t_2 \in [\alpha, \beta] \quad \left(\left(t_1 \le t_2\right) \Rightarrow \left(\text{ArcLength}\left(g \,\middle|\, \left[t_1, t_2\right]\right) = \int_{t_1}^{t_2} \sqrt{1 + \left(g'(\tau)\right)^2}\, d\tau \right) \right)$$

This can be found in "Bronstein, Semendjajew: Taschenbuch der Mathematik, Verlag Harri Deutsch, Thun und Frankfurt (Main)".

Discussion:

The example $f : \mathbb{R} \times [t_0, b] \to \mathbb{R}$ has an important property:

$$\forall t \in [t_0, b] \quad \forall y \in \mathbb{R} \quad \left(f(y,t) = 0 \quad \Rightarrow \quad (t \text{ is irrational})\right)$$

So, if you get $f(y,t) = 0$ on your computer, you made a rounding error! This means especially:

$$\frac{h}{\sqrt{1 + \left(f\left(y_i, t_i\right)\right)^2}} < h \text{ (in the new polygonal method for } f\text{)}$$

The new method is more accurate than Euler's Polygonal Method.