1. Tools

Def.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0; 1] \quad \phi(tx + (1 - t) y) \leq t\phi(x) + (1 - t) \phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex

Rem.: Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotonically increasing$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \geq 0$

2. Gamma-Function

The Gamma-Funktion $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ is for all $\alpha \in \mathbb{R}_+$ defined through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is analytically} \quad (1)$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha + 1) = \alpha \cdot \Gamma(\alpha) \quad (2)$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k + 1) = k! \quad (3)$$

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \quad (4)$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \quad (5)$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[\text{ is monotonically increasing} \quad (6)$$

3. Idea

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
We now define a mapping $\gamma :]-1;\infty[\to \mathbb{R}$ through

$$\forall u \in]-1;\infty[\quad \gamma(u) := \Gamma(u+1)$$

Then we have with (2):

$$\forall v \in]-1;\infty[\quad \gamma(v+1) = (v+1)\,\gamma(v) \tag{7}$$

In addition we have with (6):

$$\gamma \mid [1;\infty[\text{ is monotonically increasing} \tag{8}$$

Further we define for all $\alpha \in \mathbb{R}_+$ the mapping $f_\alpha : \mathbb{R}_+ \to \mathbb{R}$ through

$$f_\alpha := \sum_{n=0}^{\infty} \frac{1}{\gamma(n+\alpha)} \cdot x^{n+\alpha} = x^\alpha \cdot \left(\sum_{n=0}^{\infty} \frac{1}{\gamma(n+\alpha)} \cdot x^n \right) \tag{9}$$

We have (because of (3) and (8)) for all $\alpha \in \mathbb{R}_+$, $t \in \mathbb{R}$ and $k \in \mathbb{N}$ mit $k \geq 1$:

$$\begin{aligned}
\sum_{n=0}^{k} \left| \frac{1}{\gamma(n+\alpha)} \cdot t^n \right| &= \frac{1}{\gamma(\alpha)} + \sum_{n=1}^{k} \frac{1}{\gamma(n+\alpha)} \cdot |t|^n \leq \\
&\leq \frac{1}{\gamma(\alpha)} + \sum_{n=1}^{k} \frac{1}{\gamma(n)} \cdot |t|^n = \\
&= \frac{1}{\Gamma(\alpha+1)} + \sum_{n=1}^{k} \frac{1}{\Gamma(n+1)} \cdot |t|^n = \\
&= \frac{1}{\Gamma(\alpha+1)} + \sum_{n=1}^{k} \frac{1}{n!} \cdot |t|^n \leq \\
&\leq \frac{1}{\Gamma(\alpha+1)} + \sum_{n=0}^{k} \frac{1}{n!} \cdot |t|^n \leq \\
&\leq \frac{1}{\Gamma(\alpha+1)} + e^{|t|}
\end{aligned}$$

So (9) defines a differentiable mapping and because of (2), (7) and (9) we have for all $\alpha \in \mathbb{R}_+$:

$$
\begin{aligned}
\left(f_\alpha\right)' &= \left(\sum_{n=0}^{\infty} \frac{1}{\gamma(n+\alpha)} \cdot x^{n+\alpha}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{1}{\gamma(n+\alpha)} \cdot \left(x^{n+\alpha}\right)' = \\
&= \sum_{n=0}^{\infty} \frac{(n+\alpha)}{\gamma(n+\alpha)} \cdot x^{n+\alpha-1} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} + \sum_{n=1}^{\infty} \frac{(n+\alpha)}{\gamma(n+\alpha)} \cdot x^{n+\alpha-1} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} + \sum_{n=0}^{\infty} \frac{(n+\alpha+1)}{\gamma(n+\alpha+1)} \cdot x^{n+\alpha} = \\
&= \frac{\alpha}{\gamma(\alpha)} \cdot x^{\alpha-1} + \sum_{n=0}^{\infty} \frac{1}{\gamma(n+\alpha)} \cdot x^{n+\alpha} = \\
&= \frac{\alpha}{\Gamma(\alpha+1)} \cdot x^{\alpha-1} + f_\alpha = \\
&= \frac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} + f_\alpha
\end{aligned}
$$

Now we have proved:

$$
\forall \alpha \in \mathbb{R}_+ \quad \left(\begin{array}{l} f_\alpha \text{ is differentiable and} \\ \text{it suffices the ordinary} \\ \text{linear differential equation} \\ \left(y_\alpha\right)' - y_\alpha = \dfrac{1}{\Gamma(\alpha)} \cdot x^{\alpha-1} \text{ on } \mathbb{R}_+ \end{array} \right) \tag{10}
$$

4. Solution of the ODE

In [2] one can find the following theorem:

Theo.:

Pre.: Let J be a non-emtpy open interval of $\mathbb{R}$.
Let $g : J \to \mathbb{R}$ be a continuous mapping.
Let $h : J \to \mathbb{R}$ be a continuous mapping.
Let $\xi \in J$.
Let $\eta \in \mathbb{R}$.

Ass.: The initial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = \eta \qquad t \in J \tag{11}$$

has exactly one solution. It exists in all of J.

Rem.: Let $G : J \to \mathbb{R}$ be the antiderivative of $g : J \to \mathbb{R}$ with $G(\xi) = 0$, i. e.

$$\forall t \in J \qquad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau$$

Then the solution of the initial-value problem above is:

$$\forall t \in J \qquad y(t) = e^{-G(t)} \cdot \left(\eta + \int_{\xi}^{t} h(\tau) \cdot e^{G(\tau)}\, d\tau \right) \tag{12}$$

5. Application of the Previous Theorem

Let $\alpha \in \mathbb{R}_+$.
Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
In the specific case of section 3. is $J = \mathbb{R}_+$ and the mappings $g : J \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are defined by

$$\forall t \in J \quad g(t) := -1$$
$$\forall t \in J \quad h(t) := \frac{1}{\Gamma(\alpha)} \cdot t^{\alpha - 1}$$

We now define a mapping $T_\alpha : \mathbb{R}_+ \to \mathbb{R}$ by

$$\forall t \in J \quad T_\alpha(t) := \Gamma(\alpha) \cdot f_\alpha(t) \cdot e^{-t} \tag{13}$$

We prove finally:

T_α is a antiderivative of $x^{\alpha-1} \cdot e^{-x}$ on $\mathbb{R}_+$

i. e.

$$\forall t, \xi \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_\xi^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau \tag{14}$$

Proof of (14):
Let $\xi \in J$. The antiderivative $G : J \to \mathbb{R}$ of $g : J \to \mathbb{R}$ with $G(\xi) = 0$ has the following form:

$$\forall t \in J \quad G(t) = \int_{\xi}^{t} g(\tau)\, d\tau = \xi - t$$

Because of (10), f_α is a solution of the initial-value problem

$$y' + g(t)\, y = h(t) \qquad y(\xi) = f_\alpha(\xi) \qquad t \in J$$

it follows with (12) for any $t \in \mathbb{R}_+$:

$$f_\alpha(t) = e^{t-\xi} \cdot \left(f_\alpha(\xi) + \int_{\xi}^{t} h(\tau) \cdot e^{\xi-\tau}\, d\tau \right) =$$

$$= e^{t} \cdot \left(f_\alpha(\xi) \cdot e^{-\xi} + \int_{\xi}^{t} h(\tau) \cdot e^{-\tau}\, d\tau \right) =$$

$$= e^{t} \cdot \left(f_\alpha(\xi) \cdot e^{-\xi} + \frac{1}{\Gamma(\alpha)} \int_{\xi}^{t} \tau^{\alpha-1} \cdot e^{-\tau}\, d\tau \right)$$

This can be transformed:

$$\forall t \in J \quad \Gamma(\alpha) \cdot \left(f_\alpha(t) \cdot e^{-t} - f_\alpha(\xi) \cdot e^{-\xi} \right) = \int_{\xi}^{t} \tau^{\alpha-1} \cdot e^{-\tau}\, d\tau$$

i. e.

$$\forall t \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_{\xi}^{t} \tau^{\alpha-1} \cdot e^{-\tau}\, d\tau$$

So (14) is proved.

6. Limites

Let $\alpha \in \mathbb{R}_+$

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.

With (9) we have:

$f_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuously extendible in 0 and

$$\lim_{\xi \to 0+} f_\alpha(\xi) = 0$$

With (13) we have:

$T_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuously extendible in 0 and

$$\lim_{\xi \to 0+} T_\alpha(\xi) = 0$$

It follows with (14):

$$\forall t \in \mathbb{R}_+ \ \int_\xi^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau \text{ converges for } (\xi \to 0+)$$

and

$$\forall t \in \mathbb{R}_+ \ T_\alpha(t) = \lim_{\xi \to 0+} \int_\xi^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau = \int_0^t \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau$$

and

$$\Gamma(\alpha) = \lim_{t \to \infty} T_\alpha(t)$$

You can get an antiderivative of $x^{\alpha-1} \cdot e^{-\beta x}$ by the substitution $\tau \mapsto \beta\tau$ $\left(\beta \in \mathbb{R}_+\right)$.

7. Literature

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