1. Tools

Def.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0; 1] \quad \phi(tx + (1 - t) y) \le t\phi(x) + (1 - t)\phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex

Rem.: Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotonically increasing$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \ge 0$

2. Gamma-Function

The Gamma-Funktion $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ is for all $\alpha \in \mathbb{R}_+$ defined through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is analytically} \tag{1}$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha + 1) = \alpha \cdot \Gamma(\alpha) \tag{2}$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k + 1) = k! \tag{3}$$

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \tag{4}$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \tag{5}$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[\text{ is monotonically increasing} \tag{6}$$

We now define a mapping $\gamma :]-1; \infty[\to \mathbb{R}$ through

$$\forall u \in]-1; \infty[\quad \gamma(u) := \Gamma(u + 1)$$

Then we have with (2):

$$\forall v \in]-1; \infty[\quad \gamma(v + 1) = (v + 1)\gamma(v) \tag{7}$$

In addition we have with (6):

$$\gamma \mid [1; \infty[\text{ is monotonically increasing} \tag{8}$$

3. A Look at the sinh-Function

Let $x = \mathrm{id}_{\mathbb{R}}$.

Let $\tilde{x} = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.

Take a look at

$$\sinh = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{\substack{i=0 \\ i \text{ ungerade}}}^{\infty} \frac{x^i}{i!}$$

For $\alpha \in \mathbb{R}_+$ we define the mapping $\tilde{s}_\alpha : \mathbb{R}_+ \to \mathbb{R}$ through

$$\begin{aligned} \tilde{s}_\alpha &:= \sum_{n=0}^{\infty} \frac{\tilde{x}^{2n+1+\alpha}}{\gamma(2n+1+\alpha)} = \\ &= \tilde{x}^{\alpha} \left(\sum_{n=0}^{\infty} \frac{\tilde{x}^{2n+1}}{\gamma(2n+1+\alpha)} \right) = \qquad (9) \\ &= \sum_{\substack{i=0 \\ i \text{ ungerade}}}^{\infty} \frac{\tilde{x}^{i+\alpha}}{\gamma(i+\alpha)} \end{aligned}$$

With the theorem about the radius of convergence we have:

$\tilde{s}_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is well-defined and differentiable (10)

4. A Look at the cosh-Function

Let $x = \mathrm{id}_{\mathbb{R}}$.
Let $\tilde{x} = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
Take a look at

$$\cosh = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \sum_{\substack{i=0 \\ i \text{ gerade}}}^{\infty} \frac{x^i}{i!}$$

For $\alpha \in \mathbb{R}_+$ we define the mapping $\tilde{c}_\alpha : \mathbb{R}_+ \to \mathbb{R}$ through

$$\begin{aligned} \tilde{c}_\alpha &:= \sum_{n=0}^{\infty} \frac{\tilde{x}^{2n+\alpha}}{\gamma(2n+\alpha)} = \\ &= \tilde{x}^{\alpha} \left(\sum_{n=0}^{\infty} \frac{\tilde{x}^{2n}}{\gamma(2n+\alpha)} \right) = \\ &= \sum_{\substack{i=0 \\ i \text{ gerade}}}^{\infty} \frac{\tilde{x}^{i+\alpha}}{\gamma(i+\alpha)} \end{aligned} \tag{11}$$

With the theorem about the radius of convergence we have:

$$\tilde{c}_\alpha : \mathbb{R}_+ \to \mathbb{R} \text{ is well-defined and differentiable} \tag{12}$$

5. Differentiate

Let $\alpha \in \mathbb{R}_+$.

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.

Then we have with (2) and (7):

$$\left(\tilde{s}_\alpha\right)' = \sum_{n=0}^{\infty} \frac{\left(x^{2n+1+\alpha}\right)'}{\gamma\,(2n + 1 + \alpha)} = \\ = \sum_{n=0}^{\infty} \frac{(2n + 1 + \alpha)}{\gamma\,(2n + 1 + \alpha)}\, x^{2n+\alpha} = \\ = \sum_{n=0}^{\infty} \frac{x^{2n+\alpha}}{\gamma\,(2n + \alpha)} = \\ = \tilde{c}_\alpha$$

and

$$\left(\tilde{c}_\alpha\right)' = \sum_{n=0}^{\infty} \frac{\left(x^{2n+\alpha}\right)'}{\gamma\,(2n + \alpha)} = \\ = \sum_{n=0}^{\infty} \frac{(2n + \alpha)}{\gamma\,(2n + \alpha)}\, x^{2n-1+\alpha} = \\ = \frac{\alpha}{\gamma\,(\alpha)}\, x^{\alpha-1} + \sum_{n=1}^{\infty} \frac{(2n + \alpha)}{\gamma\,(2n + \alpha)}\, x^{2n-1+\alpha} = \\ = \frac{\alpha}{\Gamma\,(\alpha + 1)}\, x^{\alpha-1} + \sum_{n=1}^{\infty} \frac{x^{2n-1+\alpha}}{\gamma\,(2n - 1 + \alpha)} = \\ = \frac{x^{\alpha-1}}{\Gamma\,(\alpha)} + \sum_{n=0}^{\infty} \frac{x^{2n+1+\alpha}}{\gamma\,(2n + 1 + \alpha)} = \\ = \frac{x^{\alpha-1}}{\Gamma\,(\alpha)} + \tilde{s}_\alpha$$

6. Specification of the ODE

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
We have:

$$\forall \alpha \in \mathbb{R}_+ \quad \begin{pmatrix} \tilde{s}_\alpha \\ \tilde{c}_\alpha \end{pmatrix}' = \begin{pmatrix} \left(\tilde{s}_\alpha\right)' \\ \left(\tilde{c}_\alpha\right)' \end{pmatrix} = \begin{pmatrix} \tilde{c}_\alpha \\ \tilde{s}_\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}$$

i. e.

$$\forall \alpha \in \mathbb{R}_+ \quad \left(\begin{array}{l} \begin{pmatrix} \tilde{s}_\alpha \\ \tilde{c}_\alpha \end{pmatrix} \text{ is differentiable and} \\ \text{it suffices the ordinary} \\ \text{linear differential equation} \\ y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix} \text{ on } \mathbb{R}_+ \end{array} \right) \tag{13}$$

7. Solution of the ODE

In [2] one can find the following theorem:

Theo.:

Pre.: Let $n \in \mathbb{N}_+$.
Let J be a non-empty open interval of $\mathbb{R}$.
Let $A : J \to \mathrm{M}_{n \times n}(\mathbb{R})$ be a continuous mapping.
Let $b : J \to \mathbb{R}^n$ be a continuous mapping.
Let $\xi \in J$.
Let $\eta \in \mathbb{R}^n$.

Ass.: The initial-value problem

$$y' = A(t)\,y + b(t) \qquad y(\xi) = \eta \qquad t \in J \tag{14}$$

has exactly one solution. It exists in all of J.

Rem.: With [2] there exists a fundamental system $X : J \to \mathrm{GL}_n(\mathbb{R})$ of the homogeneous ODE $y' = A(t)\,y$ with $X(\xi) = E_n$. Then the solution of the initial-value problem above is:

$$\forall t \in J \qquad y(t) = X(t)\left(\eta + \int_{\xi}^{t} \left(X(\tau)\right)^{-1} b(\tau)\, d\tau\right) \tag{15}$$

8. Application of the Previous Theorem

Let $\alpha \in \mathbb{R}_+$.

Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.

In the specific case of section 6. is $J = \mathbb{R}_+$, $n = 2$ and the mappings $A : J \to \mathrm{M}_{2\times 2}(\mathbb{R})$ and $b : J \to \mathbb{R}^2$ are defined by

$$\forall t \in J \quad A(t) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\forall t \in J \quad b(t) := \begin{pmatrix} 0 \\ \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} \end{pmatrix}$$

We define 2 differentiable mappings $f : J \to \mathbb{R}^2$ and $g : J \to \mathbb{R}^2$ by

$$\forall t \in J \quad f(t) := \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix}$$

$$\forall t \in J \quad g(t) := \begin{pmatrix} \sinh(t) \\ \cosh(t) \end{pmatrix}$$

Then we have:

$$\forall t \in J \quad f'(t) = \begin{pmatrix} \sinh(t) \\ \cosh(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix} = A(t) \cdot f(t)$$

$$\forall t \in J \quad g'(t) = \begin{pmatrix} \cosh(t) \\ \sinh(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sinh(t) \\ \cosh(t) \end{pmatrix} = A(t) \cdot g(t)$$

i. e.

$$\begin{pmatrix} f : J \to \mathbb{R}^2 \text{ und } g : J \to \mathbb{R}^2 \text{ are solutions} \\ \text{of the homogeneous ODE } y' = A(t)\, y \end{pmatrix} \tag{16}$$

We define a mapping $H : J \to M_{2\times 2}(\mathbb{R})$ by

$$\forall t \in J \quad H(t) := \begin{pmatrix} f(t) & g(t) \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \tag{17}$$

Because of (16) and (17), these mapping has the properties:

$$\forall t \in J \quad \det(H(t)) = \cosh^2(t) - \sinh^2(t) = 1 \neq 0 \tag{18}$$

$$\forall t \in J \quad H(t) \in GL_2(\mathbb{R}) \tag{19}$$

$$\forall t \in J \quad (H(t))^{-1} = \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} H : J \to GL_2(\mathbb{R}) \text{ is a fundamental system} \\ \text{of the homogeneous ODE } y' = A(t)\,y \end{pmatrix} \tag{21}$$

We define a mapping $T_\alpha : J \to \mathbb{R}^2$ by

$$\begin{aligned} \forall t \in J \quad T_\alpha(t) &:= \Gamma(\alpha) \cdot (H(t))^{-1} \cdot \begin{pmatrix} \tilde{s}_\alpha(t) \\ \tilde{c}_\alpha(t) \end{pmatrix} = \\ &= \Gamma(\alpha) \cdot \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix} \cdot \begin{pmatrix} \tilde{s}_\alpha(t) \\ \tilde{c}_\alpha(t) \end{pmatrix} = \\ &= \Gamma(\alpha) \begin{pmatrix} \tilde{s}_\alpha(t)\cosh(t) - \tilde{c}_\alpha(t)\sinh(t) \\ -\tilde{s}_\alpha(t)\sinh(t) + \tilde{c}_\alpha(t)\cosh(t) \end{pmatrix} \end{aligned} \tag{22}$$

Finally we prove:

T_α is an antiderivative of $x^{\alpha-1}\begin{pmatrix} -\sinh(x) \\ \cosh(x) \end{pmatrix}$ on $\mathbb{R}_+$

respectively

$$\forall t, \xi \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_\xi^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau \tag{23}$$

Proof of (23):
Let $\xi \in J$.
We define a mapping $X : J \to \mathrm{GL}_2(\mathbb{R})$ by

$$\forall t \in J \quad X(t) := H(t) \cdot \left(H(\xi)\right)^{-1}$$

Because $\left(H(\xi)\right)^{-1}$ is constant and regular, it follows with (21) and [2]:

$$\left(\begin{array}{l} X : J \to \mathrm{GL}_2(\mathbb{R}) \text{ is a fundamental system} \\ \text{of the homogeneous ODE } y' = A(t)\, y \text{ and} \\ \ X(\xi) = E_2 \end{array}\right)$$

Because of (13), $\begin{pmatrix} \tilde{s}_\alpha \\ \tilde{c}_\alpha \end{pmatrix}$ is a solution of the inital-value problem

$$y' = A(t)\, y + b(t) \qquad y(\xi) = \begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} \qquad t \in J$$

With the theorem in section 7. and (15) we have for all $t \in J$:

$$\begin{aligned}
\begin{pmatrix} \tilde{s}_\alpha(t) \\ \tilde{c}_\alpha(t) \end{pmatrix} &= X(t)\left(\begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} + \int_\xi^t \left(X(\tau)\right)^{-1} b(\tau)\, d\tau\right) = \\
&= H(t)\left(H(\xi)\right)^{-1}\left(\begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} + \int_\xi^t \left(H(\tau)\left(H(\xi)\right)^{-1}\right)^{-1} b(\tau)\, d\tau\right) = \\
&= H(t)\left(H(\xi)\right)^{-1}\left(\begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} + \int_\xi^t H(\xi)\left(H(\tau)\right)^{-1} b(\tau)\, d\tau\right) = \\
&= H(t)\left(\left(H(\xi)\right)^{-1}\begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} + \int_\xi^t \left(H(\tau)\right)^{-1} b(\tau)\, d\tau\right)
\end{aligned}$$

But we have for all $t \in J$:

$$\left(H(t)\right)^{-1} b(t) = \frac{1}{\Gamma(\alpha)} \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix} \begin{pmatrix} 0 \\ t^{\alpha-1} \end{pmatrix} =$$

$$= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \begin{pmatrix} -\sinh(t) \\ \cosh(t) \end{pmatrix}$$

Finally, we can transform

$$\forall t \in J \quad \Gamma(\alpha) \left(\left(H(t)\right)^{-1} \begin{pmatrix} \tilde{s}_\alpha(t) \\ \tilde{c}_\alpha(t) \end{pmatrix} - \left(H(\xi)\right)^{-1} \begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} \right) =$$

$$= \int_\xi^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau$$

respectively

$$\forall t \in J \quad T_\alpha(t) - T_\alpha(\xi) = \int_\xi^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau$$

With this (23) is proved.

9. Limites

Let $\alpha \in \mathbb{R}_+$.
With (9) and (11) we have:

$$\begin{pmatrix} \tilde{s}_\alpha \\ \tilde{c}_\alpha \end{pmatrix} : \mathbb{R}_+ \to \mathbb{R}^2 \text{ is continuously extendible in 0 and}$$

$$\lim_{\xi \to 0+} \begin{pmatrix} \tilde{s}_\alpha(\xi) \\ \tilde{c}_\alpha(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

With (22) we have:

$$T_\alpha : \mathbb{R}_+ \to \mathbb{R}^2 \text{ is continuously extendible in 0 and}$$

$$\lim_{\xi \to 0+} T_\alpha(\xi) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows with (23):

$$\forall t \in \mathbb{R}_+ \quad \int_\xi^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau \text{ converges for } (\xi \to 0+)$$

and

$$\forall t \in \mathbb{R}_+ \quad T_\alpha(t) = \lim_{\xi \to 0+} \int_\xi^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau =$$

$$= \int_0^t \tau^{\alpha-1} \begin{pmatrix} -\sinh(\tau) \\ \cosh(\tau) \end{pmatrix} d\tau$$

10. Result

Let $\alpha \in \mathbb{R}_+$.
Let $x = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
Then we have:

$\Gamma(\alpha)\left(-\tilde{s}_\alpha(x)\cosh(x) + \tilde{c}_\alpha(x)\sinh(x)\right)$ is an antiderivative of $x^{\alpha-1}\sinh(x)$ on $\mathbb{R}_+$ and

$$\forall t \in \mathbb{R}_+ \quad \Gamma(\alpha)\left(-\tilde{s}_\alpha(t)\cosh(t) + \tilde{c}_\alpha(t)\sinh(t)\right) = \int_0^t \tau^{\alpha-1}\sinh(\tau)\,d\tau$$

and

$\Gamma(\alpha)\left(-\tilde{s}_\alpha(x)\sinh(x) + \tilde{c}_\alpha(x)\cosh(x)\right)$ is an antiderivative of $x^{\alpha-1}\cosh(x)$ on $\mathbb{R}_+$ and

$$\forall t \in \mathbb{R}_+ \quad \Gamma(\alpha)\left(-\tilde{s}_\alpha(t)\sinh(t) + \tilde{c}_\alpha(t)\cosh(t)\right) = \int_0^t \tau^{\alpha-1}\cosh(\tau)\,d\tau$$

You can obviously get antiderivatives of $x^{\alpha-1}\sinh(\beta x)$ and $x^{\alpha-1}\cosh(\beta x)$ by the substitution $\tau \mapsto \beta\tau$ $\left(\beta \in \mathbb{R}_+\right)$.

Because of $\forall \tau \in \mathbb{R}$ $\left(\sinh(-\tau) = -\sinh(\tau) \;\wedge\; \cosh(-\tau) = \cosh(\tau)\right)$ you can obviously get at last antiderivatives of $x^{\alpha-1}\sinh(\beta x)$ und $x^{\alpha-1}\cosh(\beta x)$ $(\beta \neq 0)$.

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