Geometry
of
Cartan’s Derivation

1. Terms

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.

1. We define $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ as the $\mathbb{R}$-vectorspace of all r-times $\mathbb{R}$-multilinear mappings $f: \underbrace{\mathbb{R}^m \times \ldots \times \mathbb{R}^m}_{r\text{-times}} \to \mathbb{R}$.

2. We define S_r as the group of all permutations $\sigma: \{1, \ldots, r\} \to \{1, \ldots, r\}$. Further we define $\mathrm{sgn}_r(\ldots)$ as the signum-function of S_r.

3. Let $f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$. We define:

$$(f \text{ is alternating}) \;:\Leftrightarrow \left(\forall \pi \in \mathrm{S}_r \quad \forall v_1, \ldots, v_r \in \mathbb{R}^m \quad f\left(v_1, \ldots, v_r\right) = \mathrm{sgn}_r(\pi)\, f\left(v_{\pi(1)}, \ldots, v_{\pi(r)}\right)\right)$$

4. We define:

$$\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) := \left\{f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) : \; f \text{ is alternating}\right\}$$

$\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is a $\mathbb{R}$-subvectorspace of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

5. Let V be a finite dimensional $\mathbb{R}$-vectorspace. Let $f : V \to V$ be a $\mathbb{R}$-linear mapping. Let U be a $\mathbb{R}$-subvectorspace of V. We define:

$$\left(f \text{ is a projection of } V \text{ onto } U\right) \quad :\Leftrightarrow \quad \left(f \circ f = f \quad \text{and} \quad f(V) = U\right)$$

Let f be a projection of V onto U. Then we have:

$$\left(\forall u \in U \quad f(u) = u\right) \quad \text{and} \quad \left(\text{kern}(f)\right) \cap U = \{0\}$$

6. Let G be an open subset of $\mathbb{R}^m$. Let $\varphi \in C^\infty(G)$. Then we define $d\varphi : G \to \mathfrak{L}^1\left(\mathbb{R}^m, \mathbb{R}\right)$ as the total differential of φ.

7. Let G be an open subset of $\mathbb{R}^m$. Then we define $A_r^m(G)$ as the set of all alternating C^∞-differential forms $\omega : G \to \mathfrak{L}_{\text{alt}}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.
Further we define $\mathfrak{d}\ldots$ as the so called Cartan's – or exterior - derivation.

2. Projection and alternating Multilinear Forms

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.

Now we define a $\mathbb{R}$ -linear mapping $\mathrm{pr}_{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}_{r,m}(\varphi)\right)\left(v_1, \dots, v_r\right) :=$$
$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\ \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

Then we have the following theorem:

Theo.:

Ass.: $\mathrm{pr}_{r,m}$ is a projection of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ onto $\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$.

Proof: In 3 Steps:

1. $\mathrm{pr}_{r,m}\left(\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)\right) \subseteq \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is true, i.e we have to prove

$$\forall f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \mathrm{pr}_{r,m}(f) \text{ is alternating} \qquad (1)$$

Proof of (1):

Let $f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

Let $v_1, \dots, v_r \in \mathbb{R}^m$.
Let $\pi \in S_r$.

We now define $w_1, \dots, w_r \in \mathbb{R}^m$ by

$$\forall i \in \{1, \dots, r\} \quad w_i := v_{\pi(i)} \tag{2}$$

Then we have:

$$\forall \kappa \in S_r \quad \forall j \in \{1, \dots, r\} \quad w_{\kappa(j)} = v_{\pi \circ \kappa(j)} \tag{3}$$

Now the following is true:

$$\begin{aligned}
&\left(\mathrm{pr}_{r,m}(f)\right)\left(v_1, \dots, v_r\right) = \\
&\quad = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\ f\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right) = \\
&\quad = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\pi \circ \sigma)\ f\left(v_{\pi \circ \sigma(1)}, \dots, v_{\pi \circ \sigma(r)}\right)
\end{aligned}$$

With (2) and (3) we have:

$$\begin{aligned}
&\left(\mathrm{pr}_{r,m}(f)\right)\left(v_1, \dots, v_r\right) = \\
&\quad = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\pi \circ \sigma)\ f\left(w_{\sigma(1)}, \dots, w_{\sigma(r)}\right) = \\
&\quad = \frac{\mathrm{sgn}_r(\pi)}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\ f\left(w_{\sigma(1)}, \dots, w_{\sigma(r)}\right) = \\
&\quad = \mathrm{sgn}_r(\pi)\left(\mathrm{pr}_{r,m}(f)\right)\left(w_1, \dots, w_r\right) = \\
&\quad = \mathrm{sgn}_r(\pi)\left(\mathrm{pr}_{r,m}(f)\right)\left(v_{\pi(1)}, \dots, v_{\pi(r)}\right)
\end{aligned}$$

2. $\mathrm{pr}_{r,m}\left(\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)\right) \supseteq \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is true, i.e because of $\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) \subseteq \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ we have to pove:

$$\forall g \in \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) \quad \mathrm{pr}_{r,m}(g) = g \tag{4}$$

Proof of (4):

Let $g \in \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$.

Let $v_1, \ldots, v_r \in \mathbb{R}^m$.
Then we have:

$$\begin{aligned}\left(\mathrm{pr}_{r,m}(g)\right)\left(v_1, \ldots, v_r\right) &= \\ &= \frac{1}{r!}\sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\; g\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right)\end{aligned} \tag{5}$$

Because $g \in \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$, the following is true:

$$\begin{aligned}\forall \sigma \in S_r \quad g\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) &= \\ &= \mathrm{sgn}_r(\sigma)\, g\left(v_1, \ldots, v_r\right)\end{aligned} \tag{6}$$

Moreover the following is true:

$$\# S_r = r! \tag{7}$$

With (5) - (7) we have:

$$\left(\mathrm{pr}_{r,m}(g)\right)\left(v_1, \ldots, v_r\right) = g\left(v_1, \ldots, v_r\right)$$

3. $\mathrm{pr}_{r,m} \circ \mathrm{pr}_{r,m} = \mathrm{pr}_{r,m}$ is true, i.e. we have to prove

$$\forall h \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \left(\mathrm{pr}_{r,m} \circ \mathrm{pr}_{r,m}\right)(h) = \mathrm{pr}_{r,m}(h)$$

But this is a consequence of (1) und (4).

3. Tools

Theo.: Formula for the Signum-Function sgn_r on S_r

Pre.: Let $r \in \mathbb{N}_+$ be with $r \geq 2$.

Ass.:

$$\forall \pi \in S_r \quad \mathrm{sgn}_r(\pi) = \prod_{1 \leq i < j \leq r} \frac{\pi(i) - \pi(j)}{i - j}$$

Rem.: Because of $S_1 = \left\{ \mathrm{id}_{\{1\}} \right\}$ the following is true:

$$\forall \pi \in S_1 \quad \mathrm{sgn}_1(\pi) = 1 = \prod_{1 \leq i < j \leq 1} \frac{\pi(i) - \pi(j)}{i - j}$$

Theo.: A Property of the Signum-Function sgn_r on S_r

Pre.: Let $r \in \mathbb{N}_+$.
Let $\pi \in S_{r+1}$.

Ass.: $\pi(r+1) = r+1 \quad \Rightarrow$
$(\pi \mid \{1, \dots, r\}) \in S_r \quad \wedge \quad \mathrm{sgn}_r (\pi \mid \{1, \dots, r\}) = \mathrm{sgn}_{r+1}(\pi)$

Proof: Let $\pi(r+1) = r+1$.
Because $\pi : \{1, \dots, r+1\} \to \{1, \dots, r+1\}$ is bijective, we have:

$$(\pi \mid \{1, \dots, r\}) : \{1, \dots, r\} \to \{1, \dots, r\} \text{ is bijective}$$

respectively

$$(\pi \mid \{1, \dots, r\}) \in S_r$$

Then we get immediately:

$$\mathrm{sgn}_{r+1}(\pi) = \prod_{1 \le i < j \le r+1} \frac{\pi(i) - \pi(j)}{i - j} =$$

$$= \left(\prod_{1 \le i < j \le r} \frac{\pi(i) - \pi(j)}{i - j} \right) \left(\prod_{1 \le i < j = r+1} \frac{\pi(i) - \pi(j)}{i - j} \right) =$$

$$= \mathrm{sgn}_r (\pi \mid \{1, \dots, r\}) \left(\prod_{i=1}^{r} \frac{\pi(i) - \pi(r+1)}{i - (r+1)} \right) =$$

$$= \mathrm{sgn}_r (\pi \mid \{1, \dots, r\}) \underbrace{\left(\prod_{i=1}^{r} \frac{\pi(i) - (r+1)}{i - (r+1)} \right)}_{=1} =$$

$$= \mathrm{sgn}_r (\pi \mid \{1, \dots, r\})$$

Def.: Let $r \in \mathbb{N}_+$.
Let $k \in \{1, \ldots, r+1\}$.
We now define $\lambda_{k,r} \in S_{r+1}$ by

$$\forall i \in \{1, \ldots, r+1\} \quad \lambda_{k,r}(i) := \begin{cases} i & i < k \ \wedge \ i < r+1 \\ i+1 & i \geq k \ \wedge \ i < r+1 \\ k & i = r+1 \end{cases}$$

Rem.: One can describe $\lambda_{k,r}$ by

$$\lambda_{k,r} = \begin{pmatrix} 1 & \ldots & k-1 & k & \ldots & r & r+1 \\ 1 & \ldots & k-1 & k+1 & \ldots & r+1 & k \end{pmatrix} \qquad (*)$$

Then the following is obvious:

$$\lambda_{k,r}(1) < \ldots < \lambda_{k,r}(r) \qquad (**)$$

Theo:

Pre.: Let $r \in \mathbb{N}_+$.
Let $k \in \{1, \ldots, r+1\}$.

Ass.: $\operatorname{sgn}_{r+1}\left(\lambda_{k,r}\right) = (-1)^{r+k-1}$

Proof: First, we have:

$$\operatorname{sgn}_{r+1}\left(\lambda_{k,r}\right) = \prod_{1 \leq i < j \leq r+1} \frac{\lambda_{k,r}(i) - \lambda_{k,r}(j)}{i - j} =$$

$$= \underbrace{\left(\prod_{1 \leq i < j \leq r} \frac{\lambda_{k,r}(i) - \lambda_{k,r}(j)}{i - j}\right)}_{>0 \text{ nach } (**)} \underbrace{\left(\prod_{i=1}^{r} \frac{\lambda_{k,r}(i) - \lambda_{k,r}(r+1)}{i - (r+1)}\right)}_{= \prod_{i=1}^{r} \frac{\lambda_{k,r}(i) - k}{i - (r+1)}}$$

We define a function $V : \mathbb{R} \setminus \{0\} \to \{1,-1\}$ by

$$\forall z \in \mathbb{R} \setminus \{0\} \quad V(z) := \begin{cases} +1 & z > 0 \\ -1 & z < 0 \end{cases}$$

Then we have to prove:

$$V\left(\prod_{i=1}^{r} \frac{\lambda_{k,r}(i)-k}{i-(r+1)}\right) = (-1)^{r+k-1} \tag{1}$$

Proof of this:

First we have obviously:

$$V\left(\prod_{i=1}^{r} i-(r+1)\right) = (-1)^{r} \tag{2}$$

By (*) the following is true:

$$\forall i \in \{1,\ldots,k-1\} \quad V\left(\lambda_{k,r}(i)-k\right) = -1 \tag{3}$$

and

$$\forall i \in \{k,\ldots,r\} \quad V\left(\lambda_{k,r}(i)-k\right) = +1 \tag{4}$$

With (2) - (4) (1) is proved (Cave $k = 1$!).

4. Known Property of Cartan's Derivation

Theo.: Invariant Description of Cartan's Derivation

Pre.: Let $m \in \mathbb{N}_+$.

Let $r \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^m$.

Let $\omega \in A_r^m(G)$.

Let $p \in G$.

Let $v_0, \dots, v_r \in \mathbb{R}^m$.

Ass.: $(\partial\omega)_p\left(v_0, \dots, v_r\right) =$

$$= \sum_{k=0}^{r} (-1)^k \left(\mathrm{d}_p\left(\omega_{\dots}\left(v_0, \dots, v_{k-1}, v_{k+1}, \dots, v_r\right)\right)\right)\left(v_k\right)$$

5. New Invariant Description of Cartan's Derivation

Theo.:

Pre.: Let $m \in \mathbb{N}_+$.

Let $r \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^m$.

Let $\omega \in A_r^m(G)$.

Let $p \in G$.

We define $\zeta \in \mathfrak{L}^{r+1}\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall w_1, \ldots, w_{r+1} \in \mathbb{R}^m \quad \zeta\left(w_1, \ldots, w_{r+1}\right) := \\ := \left(\mathrm{d}_p\left(\omega_{\ldots}\left(w_1, \ldots, w_r\right)\right)\right)\left(w_{r+1}\right)$$

Ass.:

$$\mathrm{pr}_{r+1,m}(\zeta) = \frac{(-1)^r}{r+1}(\mathfrak{d}\omega)_p$$

Proof: Let $v_1, \ldots, v_{r+1} \in \mathbb{R}^m$.

Then we have per definitionem:

$$\left(\mathrm{pr}_{r+1,m}(\zeta)\right)\left(v_1, \ldots, v_{r+1}\right) = \\ = \frac{1}{(r+1)!} \sum_{\sigma \in S_{r+1}} \mathrm{sgn}_{r+1}(\sigma)\ \zeta\left(v_{\sigma(1)}, \ldots, v_{\sigma(r+1)}\right) \qquad (1)$$

Now the following is true

$$\sum_{\sigma \in S_{r+1}} \operatorname{sgn}_{r+1}(\sigma)\ \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) =$$

$$= \sum_{k=1}^{r+1} \sum_{\substack{\sigma \in S_{r+1} \\ \sigma(r+1)=k}} \operatorname{sgn}_{r+1}(\sigma)\ \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}, v_k\right) \tag{2}$$

and (Proof of (3) later)

$$\forall k \in \{1, \dots, r+1\} \quad \forall \begin{pmatrix} \sigma \in S_{r+1} \\ \sigma(r+1) = k \end{pmatrix}$$

$$\zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) = \tag{3}$$

$$= \operatorname{sgn}_{r+1}(\sigma) \underbrace{\operatorname{sgn}_{r+1}\left(\lambda_{k,r}\right)}_{=(-1)^{r+k-1}} \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right)$$

and

$$\forall k \in \{1, \dots, r+1\} \quad \#\left\{\sigma \in S_{r+1} \ :\ \sigma(r+1) = k\right\} = r! \tag{4}$$

With (1) - (4) the following is true:

$$\left(\mathrm{pr}_{r+1,m}(\zeta)\right)\left(v_1, \dots, v_{r+1}\right) =$$

$$= \frac{(-1)^r\ r!}{(r+1)!} \sum_{k=1}^{r+1} (-1)^{k-1}\ \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right) =$$

$$= \frac{(-1)^r}{r+1} \sum_{k=1}^{r+1} (-1)^{k-1}$$

$$\left(\mathrm{d}_p\left(\omega_{\dots}\left(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_{r+1}\right)\right)\right)\left(v_k\right) =$$

$$= \frac{(-1)^r}{r+1} (\mathfrak{d}\omega)_p \left(v_1, \dots, v_{r+1}\right)$$

Now follows the proof of (3):

Let $k \in \{1, \dots, r+1\}$.
Let $\sigma \in S_{r+1}$ and be $\sigma(r+1) = k$ true.
We now define $\pi \in S_{r+1}$ by

$$\pi := \sigma^{-1} \circ \lambda_{k,r} \in S_{r+1} \tag{5}$$

and $u_1, \dots, u_r \in \mathbb{R}^m$ by

$$\forall i \in \{1, \dots, r\} \quad u_i := v_{\sigma(i)} \tag{6}$$

Because $\sigma(r+1) = k$, the following is true:

$$\pi(r+1) = r+1 \tag{7}$$

Especially we have:

$$\forall j \in \{1, \dots, r\} \quad \left(\pi(j) \in \{1, \dots, r\} \text{ and } u_{\pi(j)} = v_{\sigma\circ\pi(j)}\right) \tag{8}$$

Finally we get with (5) - (8) and $\omega \in A_r^m(G)$:

$$\begin{aligned}
&\zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) = \\
&\quad = \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}, v_k\right) = \\
&\quad = \zeta\left(u_1, \dots, u_r, v_k\right) = \\
&\quad = \operatorname{sgn}_r\left(\pi \mid \{1, \dots, r\}\right) \zeta\left(u_{\pi(1)}, \dots, u_{\pi(r)}, v_k\right) = \\
&\quad = \operatorname{sgn}_{r+1}(\pi)\, \zeta\left(v_{\sigma\circ\pi(1)}, \dots, v_{\sigma\circ\pi(r)}, v_k\right) = \\
&\quad = \operatorname{sgn}_{r+1}\left(\sigma^{-1} \circ \lambda_{k,r}\right) \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r)}, v_k\right) = \\
&\quad = \underbrace{\operatorname{sgn}_{r+1}\left(\sigma^{-1}\right)}_{=\operatorname{sgn}_{r+1}(\sigma)} \operatorname{sgn}_{r+1}\left(\lambda_{k,r}\right) \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right)
\end{aligned}$$