1. Excerpts from the classical Axiomatic Set Theory (ZFC)

1.1. General Premises

We discuss a domain $\mathfrak{M}$ of objects, which we call „sets". In the future context every letter A, B, M, X, Y and Z symbolize a set.

Let there be a two-place statement form „… is element of …" on the domain $\mathfrak{M}$, which we name $\in$. Especially we have then: For every two sets X, Y it is certain, whether $X \in Y$ is valid or $X \in Y$ is not valid (i. e. $X \notin Y$ is valid).

Further let = be a tow-place statement from on the domain $\mathfrak{M}$, i. e. for every two sets X, Y it is certain, whether $X = Y$ is valid or $X = Y$ is not valid (i. e. $X \neq Y$ is valid). In addition to this = shall have the following properties:

1. $\forall X \quad X = X$

2. $\forall X,Y \quad (X = Y \quad \Rightarrow \quad Y = X)$

3. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad Y = Z) \quad \Rightarrow \quad X = Z)$

4. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad X \in Z) \quad \Rightarrow \quad Y \in Z)$

A consequence of 1. - 4. is especially:

5. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad X \notin Z) \quad \Rightarrow \quad Y \notin Z)$

1.2. Axiom of Existence

Axiom:

Ass.: There exists a set M with the property

$$\forall X \quad X \notin M$$

1.3. Axiom of Extension

Axiom:

Ass.: For all sets A, B the following statement is valid:

$$\left(\forall X \quad \left(X \in A \quad \Leftrightarrow \quad X \in B\right)\right) \quad \Rightarrow \quad A = B$$

Rem.: 1. With 1.1. it is possible to proof:

$$\left(\forall X \quad \left(X \in A \quad \Leftrightarrow \quad X \in B\right)\right) \quad \Leftrightarrow \quad A = B$$

2. The following statement is true:

$$\left(\forall X \quad \left(X \in A \quad \Leftrightarrow \quad X \in B\right)\right) \quad \Leftrightarrow \quad \left(\forall X \quad \left(X \notin A \quad \Leftrightarrow \quad X \notin B\right)\right)$$

3. With the axiom of existence and the axiom of extension it is possible to proof:

   There exist one and only one set M with the property

   $$\forall X \quad X \notin M$$

1.4. Axiom-Scheme of Comprehension (out of date)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statement form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, whether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: There exists a set B with the property:

$$\forall X \quad (X \in B \quad \Leftrightarrow \quad P(X))$$

Rem.:

1. With the axiom schema of comprehension and the axiom of extension it is possible to proof:

   There exists one and only one set B with the property:

   $$\forall X \quad (X \in B \quad \Leftrightarrow \quad P(X))$$

   For this certain B we write $\{P(X)\}$.

2. "Russel's Antinomy":

   With this Axiom $\{X \notin X\}$ would be a set. With Russel this leads to a contradiction.

1.5. Axiom-Scheme of Comprehension (present)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statement form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, whether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: For every set A there exists a set B with the property:

$$\forall X \quad \left(X \in B \quad \Leftrightarrow \quad \left(X \in A \quad \wedge \quad P(X)\right)\right)$$

Rem.: With the axiom schema of comprehension and the axiom of extension it is possible to proof:

For every set A there exists one and only one set B with the property:

$$\forall X \quad \left(X \in B \quad \Leftrightarrow \quad \left(X \in A \quad \wedge \quad P(X)\right)\right)$$

For this certain B we write $\{X \in A: \ P(X)\}$.

1.6. Theorem

Theorem:

Ass.: There does not exist a set M with the property

$\forall X \quad X \in M$

Proof:

Supp.: There exists a set M with the property

$$\forall X \quad X \in M \tag{1}$$

Because M is a set, with the Axiom-Schema of comprehension we have (Cave! $P(X) := X \notin X$ defines a one-place statement form (see general premises)):

$$A := \{X \in M : \; X \notin X\} \text{ is a set} \tag{2}$$

Now we have by (1) with (2):

$$A \in M \tag{3}$$

Finally the following statement is valid:

$$A \in A \text{ or } A \notin A \tag{4}$$

1st case: $A \in A$ is true.

Then with the definition of A we have:

$A \notin A$

This is a contradiction!

2nd case: $A \notin A$ is true.

Then with (3) and the definition of A we have:

$A \in A$

This is a contradiction!

2. An Excerpt from an Alien-Language-ZFC

If you meet an alien ☺ in space, his ZFC might look like this chapter 2.

2.1. General Premises

We discuss a domain $\mathfrak{M}$ of objects, which we call „sets". In the future context every letter A, B, M, X, Y and Z symbolize a set.

Let there be a two-place statement form „… xyz …" on the domain $\mathfrak{M}$, which we name α. Especially we have then: For every two sets X, Y it is certain, whether $X \alpha Y$ is valid or $X \alpha Y$ is not valid (i. e. $X \not\alpha Y$ is valid).

Further let = be a tow-place statement form on the domain $\mathfrak{M}$, i. e. for every two sets X, Y it is certain, whether $X = Y$ is valid or $X = Y$ is not valid (i. e. $X \neq Y$ is valid). In addition to this = shall have the following properties:

1. $\forall X \quad X = X$

2. $\forall X,Y \quad (X = Y \quad \Rightarrow \quad Y = X)$

3. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad Y = Z) \quad \Rightarrow \quad X = Z)$

4. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad X \alpha Z) \quad \Rightarrow \quad Y \alpha Z)$

A consequence of 1. - 4. is especially:

5. $\forall X,Y,Z \quad ((X = Y \quad \wedge \quad X \not\alpha Z) \quad \Rightarrow \quad Y \not\alpha Z)$

2.2. Axiom of Existence

Axiom:

Ass.: There exists a set M with the property

$$\forall X \quad X \not\alpha M$$

2.3. Axiom of Extension

Axiom:

Ass.: For all sets A, B the following stament is valid:

$$\left(\forall X \quad \left(X\alpha A \quad \Leftrightarrow \quad X\alpha B\right)\right) \quad \Rightarrow \quad A = B$$

Rem.: 1. With 2.1. it is possible to proof:

$$\left(\forall X \quad \left(X\alpha A \quad \Leftrightarrow \quad X\alpha B\right)\right) \quad \Leftrightarrow \quad A = B$$

2. The following statement is true:

$$\left(\forall X \quad \left(X\alpha A \quad \Leftrightarrow \quad X\alpha B\right)\right) \quad \Leftrightarrow \quad \left(\forall X \quad \left(X \not\alpha A \quad \Leftrightarrow \quad X \not\alpha B\right)\right)$$

3. With the axiom of existence and the axiom of extension it is possible to proof:

There exist one and only one set M with the property

$$\forall X \quad X \not\alpha M$$

2.4. Axiom-Scheme of Comprehension (out of date)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statement form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, whether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: There exists a set B with the property:

$$\forall X \quad (X \alpha B \quad \Leftrightarrow \quad P(X))$$

Rem.: 1. With the axiom schema of comprehension and the axiom of extension it is possible to proof:

There exists one and only one set B with the property:

$$\forall X \quad (X \alpha B \quad \Leftrightarrow \quad P(X))$$

For this certain B we write $\{P(X)\}_{\alpha}$.

2. “Russel’s Antinomy”:

With this Axiom $\{X \not\alpha X\}_{\alpha}$ would be a set. With Russel this leads to a contradiction.

2.5. Axiom-Scheme of Comprehension (present)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statement form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, whether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: For every set A there exists a set B with the property:

$$\forall X \ \left(X\alpha B \ \Leftrightarrow \ \left(X\alpha A \ \wedge \ P(X)\right)\right)$$

Rem.: With the axiom schema of comprehension and the axiom of extension it is possible to proof:

For every set A there exists one and only one set B with the property:

$$\forall X \ \left(X\alpha B \ \Leftrightarrow \ \left(X\alpha A \ \wedge \ P(X)\right)\right)$$

For this certain B we write $\{X\alpha A: \ P(X)\}_{\alpha}$.

2.6. Theorem

Theorem:

Ass.: There does not exist a set M with the property

$\forall X \;\; X \alpha M$

Proof:

Supp.: There exists a set M with the property

$$\forall X \;\; X \alpha M \tag{1}$$

Because M is a set, with the Axiom-Schema of comprehension we have (Cave! $P(X) := X \not\alpha X$ defines a one-place statement form (see general premises)):

$$A := \{X \alpha M : \; X \not\alpha X\}_\alpha \text{ is a set} \tag{2}$$

Now we have by (1) with (2):

$$A \alpha M \tag{3}$$

Finally the following statement is valid:

$$A \alpha A \text{ or } A \not\alpha A \tag{4}$$

1st case: $A \alpha A$ is true.

Then with the definition of A we have:

$A \not\alpha A$

This is a contradiction!

2nd case: $A \not\alpha A$ is true.

Then with (3) and the definition of A we have:

$A \alpha A$

This is a contradiction!

3. Observations

3.1. Comparisons with ZFC and $\notin$ –ZFC

If you look at chapter 2., you cannot decide, which system of axioms (ZFC or $\notin$ –ZFC) is meant.

3.2. Extension of [2]

Every two-place statement form β on the domain $\mathfrak{M}$ causes a set theory β–ZFC. This set theory may be more difficult to interpret than ZFC or $\notin$ –ZFC.

If you look at chapter 2., you cannot decide, which system of axioms β–ZFC is meant.

3.3. Symmetry

Because of **symmetry** for every two-place statement form β on the domain $\mathfrak{M}$ there exists an alien ☺, who uses β–ZFC as its set theory.

4. Modification of 2.1.

So, if you met aliens ☺ in space, you could assume only the following theory about their ZFCs:

We discuss a domain $\mathfrak{M}$ of objects, which we call „sets". In the future context every letter A, B, M, X, Y and Z symbolize a set.

Let α be a two-place statement form on the domain $\mathfrak{M}$. Especially we have then: For every two sets X, Y it is certain, whether $X\alpha Y$ is valid or $X\alpha Y$ is not valid (i. e. $X\not\alpha Y$ is valid).

Further let = be a tow-place statement form on the domain $\mathfrak{M}$, i. e. for every two sets X, Y it is certain, whether $X=Y$ is valid or $X=Y$ is not valid (i. e. $X\neq Y$ is valid). In addition to this = shall have the following properties:

1. $\forall X \quad X=X$

2. $\forall X,Y \quad (X=Y \quad \Rightarrow \quad Y=X)$

3. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad Y=Z) \quad \Rightarrow \quad X=Z)$

4. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad X\alpha Z) \quad \Rightarrow \quad Y\alpha Z)$

A consequence of 1. - 4. is especially:

5. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad X\not\alpha Z) \quad \Rightarrow \quad Y\not\alpha Z)$

This leads to a contradiction (like in [2])!

5. Index of Literature

[1] Lectures in Mathematics 1987 - 2002
University of Cologne (Germany)

[2] „The End of ZFC"
WWW.Reinbothe.DE

[3] Introduction to Axiomatic Set Theory
Graduate Texts in Mathematics
G. Takeuti, W. M. Zaring
Springer-Verlag