A New Kind Of Power Series And Two Related Transformations

By Christian Reinbothe

http://WWW.Reinbothe.DE/english/mathPreprints.htm

Abstract

Consider the problem of getting an antiderivative of $x^{\alpha - 1} F(x)$ $\left(\alpha \in \mathbb{R}_+\right)$ for a given standard power series $F(x) = \sum_{n=0}^{\infty} a_n x^n$. The solution is the transformation of a standard power series in a new kind of power series:

$$\sum_{n=0}^{\infty} a_n x^n \quad \underset{\alpha}{\mapsto} \quad \sum_{n=0}^{\infty} \frac{1}{n + \alpha} a_n x^{n+\alpha} \tag{T1}$$

Now take a look at another standard power series G. With $\gamma(z) := \Gamma(z + 1)$ we have:

$$G(x) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{\gamma(n)} x^n$$

So there is a second transformation of a standard power series in the new kind of power series:

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n \quad \underset{\alpha}{\mapsto} \quad \sum_{n=0}^{\infty} \frac{b_n}{\gamma(n + \alpha)} x^{n+\alpha} \tag{T2}$$

Differentiating the transformation (T2) provides you ideally with a benign linear ODE. Then a comparison of the transformation (T1) with the terms of the explicit solution of the ODE leads to an interesting equation. You can apply this method on every standard power series, for which there is a “good” inner connection of the b_n.

In the case of the here considered examples the resulting equations compare two different expansion series for an antiderivative of $x^{\alpha - 1} F(x)$ respectively $x^{\alpha} F(x)$.

Consideration Of $F(x) = e^{x}$ And $G(x) = e^{-x}$

For every $\alpha, t \in \mathbb{R}_{+}$ the following is true:

$$\sum_{n=0}^{\infty} \left(\left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - (-1)^n \, e^t \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{t^{n+\alpha}}{n+\alpha} = 0$$

The proof of this is elementary.

Consideration Of $F(x) = e^{-x}$ And $G(x) = e^{x}$

For every $\alpha, t \in \mathbb{R}_{+}$ the following is true:

$$\sum_{n=0}^{\infty} \left((-1)^n \left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - e^{-t} \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{t^{n+\alpha}}{n+\alpha} = 0$$

The proof of this is elementary.

Consideration Of $F(x) = 1 / (1 + x)$ And $G(x) = 1 / (1 - x)$

For every $\alpha \in \mathbb{R}_{+}$ and every $t \in \left]0; 1\right[$ the following is true:

$$\sum_{n=0}^{\infty} \left((-1)^n \left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - \frac{1}{(1+t)^{n+1}} \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{n!}{n+\alpha} t^{n+\alpha} = 0$$

Consideration Of $F(x) = 1 / (1 - x)$ And $G(x) = 1 / (1 + x)$

For every $\alpha \in \mathbb{R}_{+}$ and every $t \in \left]0; \frac{1}{2}\right[$ the following is true:

$$\sum_{n=0}^{\infty} \left(\left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - \frac{(-1)^n}{(1-t)^{n+1}} \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{n!}{n+\alpha} t^{n+\alpha} = 0$$

Consideration Of $F(x) = 1/(1+x)$ And $G(x) = -\ln(1-x)$

For every $\alpha \in \mathbb{R}_+$ and every $t \in]0;1[$ the following is true:

$$\sum_{n=1}^{\infty}\left((-1)^{n-1}\left(\prod_{i=1}^{n-1}\frac{1}{i}\right)-\frac{1}{(1+t)^n}\left(\prod_{i=1}^{n-1}\frac{1}{i+\alpha}\right)\right)\frac{(n-1)!}{n+\alpha}\,t^{n+\alpha}=0$$

Consideration Of $F(x) = 1/(1-x)$ And $G(x) = \ln(1+x)$

For every $\alpha \in \mathbb{R}_+$ and every $t \in \left]0;\frac{1}{2}\right[$ the following is true:

$$\sum_{n=1}^{\infty}\left(\left(\prod_{i=1}^{n-1}\frac{1}{i}\right)-\frac{(-1)^{n-1}}{(1-t)^n}\left(\prod_{i=1}^{n-1}\frac{1}{i+\alpha}\right)\right)\frac{(n-1)!}{n+\alpha}\,t^{n+\alpha}=0$$

Conclusion

You can summarize the equations from above in pairs. Then you get:

For every $\alpha \in \mathbb{R}_+$ and $t \in \mathbb{R}$ the following is true:

$$\sum_{n=0}^{\infty} \left((-1)^n \left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - e^{-t} \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{t^n}{n+\alpha} = 0 \tag{E1}$$

For every $\alpha \in \mathbb{R}_+$ and every $t \in \left] -\frac{1}{2}; 1 \right[$ the following is true:

$$\sum_{n=0}^{\infty} \left((-1)^n \left(\prod_{i=0}^{n-1} \frac{1}{i+1} \right) - \frac{1}{(1+t)^{n+1}} \left(\prod_{i=0}^{n-1} \frac{1}{i+\alpha} \right) \right) \frac{n!}{n+\alpha} t^n = 0 \tag{E2}$$

For every $\tilde{\alpha} \in \mathbb{R}_+$ and every $t \in \left] -\frac{1}{2}; 1 \right[$ the following is true:

$$\sum_{n=1}^{\infty} \left((-1)^{n-1} \left(\prod_{i=1}^{n-1} \frac{1}{i} \right) - \frac{1}{(1+t)^{n}} \left(\prod_{i=1}^{n-1} \frac{1}{i+\tilde{\alpha}} \right) \right) \frac{(n-1)!}{n+\tilde{\alpha}} t^n = 0 \tag{E3}$$

The equation (E3) is a consequence of the equation (E2) with $\alpha := \tilde{\alpha} + 1$.

Outlook

You can use this method successfully on $G(x) = \sin(x)$, $G(x) = \cos(x)$, $G(x) = \sinh(x)$ and $G(x) = \cosh(x)$.

A question is, wether the domain for t in equation (E1) respectively (E2) is greater than $\mathbb{R}$ respectively $\left] -\frac{1}{2}; 1 \right[$.