Radius of Convergence

Theo.:

Pre.: Let $x = \mathrm{id}_{\mathbb{R}}$.
Let $\tilde{x} = \left(\mathrm{id}_{\mathbb{R}}\right) \mid \mathbb{R}_+$.
Let $\alpha \in \mathbb{R}_+$.
Let $\left(a_i\right)_{i \in \mathbb{N}_0}$ be a sequence in $\mathbb{R}$.

Ass..: $\left(\sum_{n=0}^{k} \frac{a_n}{n!} \cdot x^n\right)_{k \in \mathbb{N}_0}$ and $\left(\sum_{n=0}^{k} \frac{a_n}{\gamma\left(n + \alpha\right)} \cdot \tilde{x}^{n+\alpha}\right)_{k \in \mathbb{N}_0}$
have the same radius of convergence

Rem.: Let $R \in [0;\infty]$.

R is the radius of convergence of

$$\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot \tilde{x}^{n+\alpha}\right)_{k\in\mathbb{N}_0}$$

, iff for all $t \in [0;\infty[$

$$\left(t < R \quad \Rightarrow \quad \left(\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot t^{n+\alpha}\right)_{k\in\mathbb{N}_0} \text{ is absolutely convergent} \right)\right)$$

and

$$\left(t > R \quad \Rightarrow \quad \left(\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot t^{n+\alpha}\right)_{k\in\mathbb{N}_0} \text{ is divergent} \right)\right)$$

Rem.: Let $R \in [0;\infty[$.

Let R be the radius of convergence of $\left(\sum_{n=0}^{k} \frac{a_n}{n!} \cdot x^n\right)_{k\in\mathbb{N}_0}$.

With the methods of the following proof one can show:

$$\left(\left(\sum_{n=0}^{k} \frac{a_n}{n!} \cdot R^n\right)_{k\in\mathbb{N}_0} \text{ is absolutely convergent} \right) \Rightarrow$$

$$\left(\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot R^{n+\alpha}\right)_{k\in\mathbb{N}_0} \text{ is absolutely convergent} \right)$$

Proof: Let $\rho_0 \in [0;\infty]$ be the radius of convergence of $\left(\sum_{n=0}^{k} \frac{a_n}{n!} \cdot x^n\right)_{k\in\mathbb{N}_0}$. Let $\rho_\alpha \in [0;\infty]$ be the radius of convergence of $\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot \tilde{x}^{n+\alpha}\right)_{k\in\mathbb{N}_0}$.

We have to prove:

$$\rho_0 = \rho_\alpha$$

Hypo.: $\rho_0 \neq \rho_\alpha$

1. case: $\rho_0 > \rho_\alpha$

There is $t \in \mathbb{R}_+$ with $\rho_0 > t > \rho_\alpha$. Then we have:

$$\left(\sum_{n=0}^{k} \frac{a_n}{n\,!} \cdot t^n \right)_{k \in \mathbb{N}_0} \qquad \text{is absolutely convergent}$$

and

$$\left(\sum_{n=0}^{k} \frac{a_n}{\gamma\,(n\,+\,\alpha)} \cdot t^{n+\alpha} \right)_{k \in \mathbb{N}_0} \qquad \text{is divergent}$$

On the other hand we have because of (3) and (8) for all $k \in \mathbb{N}_0$ with $k \geq 1$:

$$\sum_{n=0}^{k} \left| \frac{a_n}{\gamma(n+\alpha)} \cdot t^{n+\alpha} \right| =$$

$$= |t|^\alpha \left(\frac{|a_0|}{\gamma\,(\alpha)} + \sum_{n=1}^{k} \frac{|a_n|}{\gamma(n+\alpha)} \cdot |t|^n \right) \leq$$

$$\leq |t|^\alpha \left(\frac{|a_0|}{\gamma\,(\alpha)} + \sum_{n=1}^{k} \frac{|a_n|}{\gamma(n)} \cdot |t|^n \right) =$$

$$= |t|^\alpha \left(\frac{|a_0|}{\Gamma\,(\alpha\,+\,1)} + \sum_{n=1}^{k} \frac{|a_n|}{\Gamma(n+1)} \cdot |t|^n \right) =$$

$$= |t|^\alpha \left(\frac{|a_0|}{\Gamma\,(\alpha\,+\,1)} + \sum_{n=1}^{k} \frac{|a_n|}{n\,!} \cdot |t|^n \right) \leq$$

$$\leq |t|^\alpha \left(\frac{|a_0|}{\Gamma\,(\alpha\,+\,1)} + \sum_{n=0}^{k} \frac{|a_n|}{n\,!} \cdot |t|^n \right)$$

This is a contradiction!

2. case: $\rho_0 < \rho_\alpha$

Because $\alpha \in \mathbb{R}_+$, there exists $l \in \mathbb{N}_+$ with $l - 1 \le \alpha < l$. Then we have with (8):

$$\forall n \in \mathbb{N}_+ \qquad \frac{1}{(n+l)!} = \frac{1}{\gamma(n+l)} \le \frac{1}{\gamma(n+\alpha)} \qquad (*)$$

If one differentiate the power series $\left(\sum_{n=0}^{k} \frac{a_n}{(n+l)!} \cdot x^{n+l}\right)_{k \in \mathbb{N}_0}$ l-times, he gets the power series $\left(\sum_{n=0}^{k} \frac{a_n}{n!} \cdot x^{n}\right)_{k \in \mathbb{N}_0}$. With Analysis we have for the radius of convergence $R \in [0; \infty]$ of $\left(\sum_{n=0}^{k} \frac{a_n}{(n+l)!} \cdot x^{n+l}\right)_{k \in \mathbb{N}_0}$

$$R \le \rho_0 < \rho_\alpha$$

Ergo there exists $t \in \mathbb{R}_+$ with $R < t < \rho_\alpha$. Then we have:

$$\left(\sum_{n=0}^{k} \frac{a_n}{(n+l)!} \cdot t^{n+l}\right)_{k \in \mathbb{N}_0} \quad \text{is divergent}$$

and

$$\left(\sum_{n=0}^{k} \frac{a_n}{\gamma(n+\alpha)} \cdot t^{n+\alpha}\right)_{k \in \mathbb{N}_0} \quad \text{is absolutely convergent}$$

On the other hand we have because of (*) for all $k \in \mathbb{N}_0$ with $k \geq 1$:

$$\sum_{n=0}^{k} \left| \frac{a_n}{(n+1)!} \cdot t^{n+1} \right| = \sum_{n=0}^{k} \frac{|a_n|}{(n+1)!} \cdot |t|^{n+1} =$$

$$= \frac{|a_0|}{1!} \cdot |t|^1 + \sum_{n=1}^{k} \frac{|a_n|}{(n+1)!} \cdot |t|^{n+1} \leq$$

$$\leq \frac{|a_0|}{1!} \cdot |t|^1 + \sum_{n=1}^{k} \frac{|a_n|}{\gamma(n+\alpha)} \cdot |t|^{n+1} \leq$$

$$\leq \frac{|a_0|}{1!} \cdot |t|^1 + \sum_{n=0}^{k} \frac{|a_n|}{\gamma(n+\alpha)} \cdot |t|^{n+1} =$$

$$= \frac{|a_0|}{1!} \cdot |t|^1 + |t|^{1-\alpha} \cdot \left(\sum_{n=0}^{k} \frac{|a_n|}{\gamma(n+\alpha)} \cdot |t|^{n+\alpha} \right)$$

This is a contradiction!