Several
New
Antiderivatives

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1. An Antiderivative Of $e^{(\alpha(x^m))}$

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
We know:

$$e^{\left(\alpha x^m\right)} = \sum_{i=0}^{\infty} \frac{1}{i!} \left(\alpha x^m\right)^i =$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\alpha^i x^{mi}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{1}{i!} \alpha^i x^{mi}$$

So we define a function $E_1 : \mathbb{R} \to \mathbb{R}$ through

$$E_1(x) := \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\alpha^i}{mi+1} x^{mi+1}$$

Then we have obviously:

$E_1 : \mathbb{R} \to \mathbb{R}$ is C^∞

$E_1 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $e^{\left(\alpha x^m\right)}$

2. An Antiderivative Of $e\hat{}(\alpha(x\hat{}m)+\beta(x\hat{}n))$

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
Let $\beta \in \mathbb{R}$ with $\beta \neq 0$.
Let $n \in \mathbb{N}_+$.
We know:

$$\begin{aligned} e^{\left(\alpha x^m + \beta x^n\right)} &= \sum_{i=0}^{\infty} \frac{1}{i\,!} \cdot \left(\alpha x^m + \beta x^n\right)^i = \\ &= \sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\sum_{j=0}^{i} \binom{i}{j} \left(\alpha x^m\right)^j \left(\beta x^n\right)^{i-j} \right) = \\ &= \sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\sum_{j=0}^{i} \binom{i}{j} \alpha^j x^{mj} \beta^{i-j} x^{n(i-j)} \right) = \\ &= \sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\sum_{j=0}^{i} \binom{i}{j} \alpha^j \beta^{i-j} x^{mj+n(i-j)} \right) \end{aligned}$$

So we define a function $E_2 : \mathbb{R} \to \mathbb{R}$ through

$$E_2(x) := \sum_{i=0}^{\infty} \frac{1}{i\,!} \left(\sum_{j=0}^{i} \binom{i}{j} \frac{\alpha^j \beta^{i-j}}{mj + n(i-j) + 1} x^{mj+n(i-j)+1} \right)$$

Then we have obviously:

$E_2 : \mathbb{R} \to \mathbb{R}$ is C^∞

$E_2 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $e^{\left(\alpha x^m + \beta x^n\right)}$

3. An Antiderivative Of $\sin(\alpha(x\text{^}m))$

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
We know:

$$\sin\left(\alpha x^m\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i + 1)!} \left(\alpha x^m\right)^{2i+1} =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i + 1)!} \left(\alpha^{2i+1} x^{2mi+m}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i + 1)!} \alpha^{2i+1} x^{2mi+m}$$

So we define a function $E_3 : \mathbb{R} \to \mathbb{R}$ through

$$E_3(x) := \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i + 1)!} \frac{\alpha^{2i+1}}{2mi + m + 1} x^{2mi+m+1}$$

Then we have obviously:

$E_3 : \mathbb{R} \to \mathbb{R}$ is C^∞

$E_3 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $\sin\left(\alpha x^m\right)$

4. An Antiderivative Of sin(α(x^m)+β(x^n))

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
Let $\beta \in \mathbb{R}$ with $\beta \neq 0$.
Let $n \in \mathbb{N}_+$.
We know:

$$\sin\left(\alpha x^m + \beta x^n\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}\left(\alpha x^m + \beta x^n\right)^{2i+1} =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}\left(\sum_{j=0}^{2i+1} \binom{2i+1}{j}\left(\alpha x^m\right)^j \left(\beta x^n\right)^{2i+1-j}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}\left(\sum_{j=0}^{2i+1} \binom{2i+1}{j}\alpha^j x^{mj} \beta^{2i+1-j} x^{2ni+n-nj}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}\left(\sum_{j=0}^{2i+1} \binom{2i+1}{j}\alpha^j \beta^{2i+1-j} x^{mj+2ni+n-nj}\right)$$

So we define a function $E_4 : \mathbb{R} \to \mathbb{R}$ through

$$E_4(x) := \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}\left(\sum_{j=0}^{2i+1} \frac{\binom{2i+1}{j}\alpha^j \beta^{2i+1-j}}{mj+2ni+n-nj+1} x^{mj+2ni+n-nj+1}\right)$$

Then we have obviously:

$E_4 : \mathbb{R} \to \mathbb{R}$ is C^∞

$E_4 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $\sin\left(\alpha x^m + \beta x^n\right)$

5. An Antiderivative Of cos(α(x^m))

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
We know:

$$\cos\left(\alpha x^m\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\alpha x^m\right)^{2i} =$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\alpha^{2i} x^{2mi}\right) =$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\,\alpha^{2i} x^{2mi}$$

So we define a function $E_5 : \mathbb{R} \to \mathbb{R}$ through

$$E_5(x) := \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\,\frac{\alpha^{2i}}{2mi+1} x^{2mi+1}$$

Then we have obviously:

$E_5 : \mathbb{R} \to \mathbb{R}$ is C^{∞}

$E_5 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $\cos\left(\alpha x^m\right)$

6. An Antiderivative Of cos(α(x^m)+β(x^n))

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
Let $\beta \in \mathbb{R}$ with $\beta \neq 0$.
Let $n \in \mathbb{N}_+$.
We know:

$$\cos\left(\alpha x^m + \beta x^n\right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\alpha x^m + \beta x^n\right)^{2i} =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\sum_{j=0}^{2i} \binom{2i}{j}\left(\alpha x^m\right)^j \left(\beta x^n\right)^{2i-j}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\sum_{j=0}^{2i} \binom{2i}{j}\alpha^j x^{mj}\beta^{2i-j}x^{2ni-nj}\right) =$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\sum_{j=0}^{2i} \binom{2i}{j}\alpha^j \beta^{2i-j}x^{mj+2ni-nj}\right)$$

So we define a function $E_6 : \mathbb{R} \to \mathbb{R}$ through

$$E_6(x) := \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!}\left(\sum_{j=0}^{2i} \frac{\binom{2i}{j}\alpha^j\beta^{2i-j}}{mj + 2ni - nj + 1} x^{mj+2ni-nj+1}\right)$$

Then we have obviously:

$E_6 : \mathbb{R} \to \mathbb{R}$ is C^∞

$E_6 : \mathbb{R} \to \mathbb{R}$ is an antiderivative of $\cos\left(\alpha x^m + \beta x^n\right)$

7. More Antiderivatives With sinh And cosh

We know:

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right) = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!}$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right) = \sum_{i=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

In this case we can apply 1. and 2.

8. More Antiderivatives With tan And tanh

We know from “Bronstein” (ISBN: 3 87144 492 8):

$$\tan(x) = \sum_{i=1}^{\infty} \frac{2^{2i}\left(2^{2i} - 1\right) B_i}{(2i)!} x^{2i-1} \qquad \left(|x| < \frac{\pi}{2}\right)$$

$$\cot(x) = \sum_{i=0}^{\infty} \frac{2^{2i} B_i}{(2i)!} x^{2i-1} \qquad \left(0 < |x| < \pi\right)$$

$$\tanh(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}\, 2^{2i}\left(2^{2i} - 1\right) B_i}{(2i)!} x^{2i-1} \qquad \left(|x| < \frac{\pi}{2}\right)$$

$$\coth(x) = \sum_{i=0}^{\infty} \frac{(-1)^{i+1}\, 2^{2i} B_i}{(2i)!} x^{2i-1} \qquad \left(0 < |x| < \pi\right)$$

$\left(B_i\right)_{i \in \mathbb{N}}$ are the Bernoulli-Numbers

Cave!: Please verify this power series!

With this you can get new antiderivatives for tan and tanh, **but not for cot and coth.** You must compute the radius of convergence of these antiderivatives.

9. Observation 1

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
Let $\left(a_i\right)_{i \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.
Then there is $\rho \in [0; \infty]$ and a power series $f :]-\rho; \rho[\to \mathbb{R}$ with:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$f :]-\rho; \rho[\to \mathbb{R} \text{ is } C^\infty$$

Then we have:

$$f\left(\alpha x^m\right) = \sum_{i=0}^{\infty} a_i \left(\alpha x^m\right)^i = \sum_{i=0}^{\infty} a_i \alpha^i x^{mi}$$

Then there is $\tilde{\rho} \in [0; \infty]$ and a power series $\tilde{f} : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R}$ with:

$$\rho < \infty \quad \Rightarrow \quad \tilde{\rho} \geq \left(\frac{\rho}{|\alpha|}\right)^{\frac{1}{m}}$$

$$\tilde{f}(x) = f\left(\alpha x^m\right)$$

$$\tilde{f} : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R} \text{ is } C^\infty$$

So we define a function $\tilde{F}_1 : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R}$ through

$$\tilde{F}_1(x) = \sum_{i=0}^{\infty} \frac{a_i \alpha^i}{mi + 1} x^{mi+1}$$

Then we have obviously:

$\tilde{F}_1 : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R}$ is a power series

$\tilde{F}_1 : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R}$ is C^∞

$\tilde{F}_1 : \left]-\tilde{\rho}; \tilde{\rho}\right[\to \mathbb{R}$ is an antiderivative of $f\left(\alpha x^m\right)$

10. Observation 2

Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$.
Let $m \in \mathbb{N}_+$.
Let $\beta \in \mathbb{R}$ with $\beta \neq 0$.
Let $n \in \mathbb{N}_+$.
Let $\left(a_i\right)_{i \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.
Then there is $\rho \in [0; \infty]$ and a power series $f :]-\rho; \rho[\to \mathbb{R}$ with:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

$$f :]-\rho; \rho[\to \mathbb{R} \text{ is } C^{\infty}$$

Then we have:

$$\begin{aligned}
f\left(\alpha x^m + \beta x^n\right) &= \sum_{i=0}^{\infty} a_i \left(\alpha x^m + \beta x^n\right)^i = \\
&= \sum_{i=0}^{\infty} a_i \left(\sum_{j=0}^{i} \binom{i}{j} \left(\alpha x^m\right)^j \left(\beta x^n\right)^{i-j} \right) = \\
&= \sum_{i=0}^{\infty} a_i \left(\sum_{j=0}^{i} \binom{i}{j} \alpha^j x^{mj} \beta^{i-j} x^{n(i-j)} \right) = \\
&= \sum_{i=0}^{\infty} a_i \left(\sum_{j=0}^{i} \binom{i}{j} \alpha^j \beta^{i-j} x^{mj+n(i-j)} \right)
\end{aligned}$$

Then there is $\tilde{\rho} \in [0; \infty]$ and a power series $\tilde{f} :]-\tilde{\rho}; \tilde{\rho}[\to \mathbb{R}$ with:

$$\tilde{\rho} = \sup\left\{ r \in \mathbb{R} : \quad \left|\alpha r^m + \beta r^n\right| \leq \rho \right\}$$

$$\tilde{f}(x) = f\left(\alpha x^m + \beta x^n\right)$$

$$\tilde{f} :]-\tilde{\rho}; \tilde{\rho}[\to \mathbb{R} \text{ is } C^{\infty}$$

So we define a function $\widetilde{F}_2 : \left]-\widetilde{\rho}; \widetilde{\rho}\right[\to \mathbb{R}$ through

$$\widetilde{F}_2(x) = \sum_{i=0}^{\infty} a_i \left(\sum_{j=0}^{i} \binom{i}{j} \frac{\alpha^j \beta^{i-j}}{mj + n(i-j) + 1} x^{mj+n(i-j)+1} \right)$$

Then we have obviously:

$\widetilde{F}_2 : \left]-\widetilde{\rho}; \widetilde{\rho}\right[\to \mathbb{R}$ is a power series

$\widetilde{F}_2 : \left]-\widetilde{\rho}; \widetilde{\rho}\right[\to \mathbb{R}$ is C^{∞}

$\widetilde{F}_2 : \left]-\widetilde{\rho}; \widetilde{\rho}\right[\to \mathbb{R}$ is an antiderivative of $f\left(\alpha x^m + \beta x^n\right)$