T=0
explained

1. Symmetry of the kth Differential

Theorem:

Pre.: Let $k \in \mathbb{N}_+$ with $k \geq 2$.
Let $n \in \mathbb{N}_+$.
Let G be an open subset of $\mathbb{R}^n$.
Let $\varphi : G \to \mathbb{R}$ be a mapping.
Let φ be k-times differentiable.

Ass.: $\forall p \in G \quad \mathrm{d}_p^k\varphi : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \to \mathbb{R}$ is symmetric

2. A Special Case of Cartan's Derivation

Theorem:

Pre.: Let $n \in \mathbb{N}_+$.
Let G be an open subset of $\mathbb{R}^n$.
Let $V : G \to \mathbb{R}^n$ be a continuous differentiable mapping.
Let $\omega : G \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ be defined as $\omega := \sum_{i=1}^{n} V_i \cdot \mathrm{d}x_i$
(espacially $\omega : G \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ is continuous differentiable).

Ass.: $\forall p \in G \quad \left(\mathfrak{d}_p\omega = 0 \quad \Leftrightarrow \quad \mathrm{d}_pV \text{ is self-adjoint}\right)$

Rem.:

1. $\mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ is the $\mathbb{R}$-vector-space of all $\mathbb{R}$-linear-forms $\mathbb{R}^n \to \mathbb{R}$.
2. ω is a so called C^1-differential form of degree 1.
3. $\mathfrak{d}\ldots$ is Cartan's derivation. If $n = 3$, then the following is true:
   $\forall p \in G \quad \left(\left(\mathfrak{d}_p\omega = 0\right) \quad \Leftrightarrow \quad \left(\mathrm{curl}_p\,(V) = 0\right)\right).$

3. Vector Potential

Theo.:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^n$.

Ass.: 1. Let $\varphi \in C^2(G)$.
Then the follwing is true:
$\left(\forall p \in G \quad \left(d_p\left(\operatorname{grad}(\varphi)\right) \text{ is self-adjoint}\right)\right)$

2. Be G star-shaped.
Be $k \in \mathbb{N}_+ \cup \{\infty\}$.
Let $V : G \to \mathbb{R}^n$ be a k-times continuous differentiable mapping.
Then the following is true:
$\left(\forall p \in G \quad \left(d_p V \text{ is self-adjoint}\right)\right) \quad \Rightarrow$
$\exists \varphi \in C^{k+1}(G) \quad V = \operatorname{grad}(\varphi)$

4. The Inversal of 1 (k=2)

Theorem:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open, star-shaped subset of $\mathbb{R}^n$.

Let $\alpha : G \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ be a continuous differentiable mapping.

Let $\beta : G \to \mathfrak{L}^2\left(\mathbb{R}^n, \mathbb{R}\right)$ be defined through

$\forall p \in G \quad \forall v, w \in \mathbb{R}^n \quad \left(\beta_p\right)(v, w) := \left(\left(\mathrm{d}_p\alpha\right)(v)\right)(w)$

Ass.: $\left(\forall p \in G \quad \left(\beta_p \text{ is symmetric}\right)\right) \Rightarrow$

$\begin{pmatrix} \text{There is } \varphi \in C^2(G) \text{ with} \\ \mathrm{d}\varphi = \alpha \text{ and } \mathrm{d}^2\varphi = \beta \end{pmatrix}$

Rem.: $\mathfrak{L}^2\left(\mathbb{R}^n, \mathbb{R}\right) = \left\{b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ is } \mathbb{R}\text{-bilinear}\right\}$ is a $\mathbb{R}$-vector-space.

Proof: Let $\mathfrak{E} := \left(e_1, \ldots, e_n\right)$ be the standard base of $\mathbb{R}^n$.

Let $\mathfrak{X} := \left(x_1, \ldots, x_n\right)$ be the base of $\mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ dual to $\mathfrak{E}$.

Let the mapping $V : G \to \mathbb{R}^n$ be defined as

$$\forall p \in G \quad V(p) := \begin{pmatrix} \alpha_p\left(e_1\right) \\ \vdots \\ \alpha_p\left(e_n\right) \end{pmatrix} \tag{1}$$

Then the following statements are valid:

$$V : G \to \mathbb{R}^n \text{ is continuous differentiable} \tag{2}$$

and

$$\forall p \in G \quad \mathrm{d}_p V = \begin{pmatrix} \mathrm{d}_p\left(\alpha_{\ldots}\left(e_1\right)\right) \\ \vdots \\ \mathrm{d}_p\left(\alpha_{\ldots}\left(e_n\right)\right) \end{pmatrix} = \begin{pmatrix} \left(\mathrm{d}_p\alpha\right)\left(e_1\right) \\ \vdots \\ \left(\mathrm{d}_p\alpha\right)\left(e_n\right) \end{pmatrix} \tag{3}$$

Then the following statement is true:

$$\forall j \in \{1, \dots, n\} \;\; \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \frac{\partial V}{\partial x_j} \tag{4}$$

A consequence of (3) is

$$\forall j \in \{1, \dots, n\} \;\; \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \begin{pmatrix} \left(\mathrm{d}_p \alpha\right)\left(e_1\right) \\ \vdots \\ \left(\mathrm{d}_p \alpha\right)\left(e_n\right) \end{pmatrix} \left(e_j\right)$$

Then it follows by premise:

$$\forall j \in \{1, \dots, n\} \;\; \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \begin{pmatrix} \left(\beta_p\right)\left(e_1, e_j\right) \\ \vdots \\ \left(\beta_p\right)\left(e_n, e_j\right) \end{pmatrix} \tag{5}$$

Because $\left(\forall p \in G \;\; \left(\beta_p \text{ is symmetric}\right)\right)$, a consequence of (4) and (5) is

$$\left(\forall p \in G \;\; \left(\mathrm{d}_p V \text{ is self-adjoint}\right)\right) \tag{6}$$

Because of (2), (6) and theroem 3.2., there exists $\varphi \in C^2(G)$ with

$$V = \mathrm{grad}(\varphi) \tag{7}$$

Because of (1) and (7), for all $\forall i \in \{1, \dots, n\}$ and all $\forall p \in G$ the following statement is valid:

$$\left(\mathrm{d}_p \varphi\right)\left(e_i\right) = \frac{\partial \varphi}{\partial x_i}(p) = \left(\mathrm{grad}_p(\varphi)\right)_i = V_i(p) = \alpha_p\left(e_i\right)$$

respectively

$$\mathrm{d}\varphi = \alpha \tag{8}$$

Then it follows for all $\forall i, j \in \{1, \dots, n\}$ and all $\forall p \in G$

$$d_p^2\varphi\left(e_i, e_j\right) = \left(d_p\left(\left(d_{\dots}\varphi\right)\left(e_i\right)\right)\right)\left(e_j\right) = \left(d_p\left(\left(\alpha_{\dots}\right)\left(e_i\right)\right)\right)\left(e_j\right)$$

respectively

$$d_p^2\varphi\left(e_i, e_j\right) = \left(\left(d_p(\alpha)\right)\left(e_i\right)\right)\left(e_j\right)$$

respectively by premise

$$d_p^2\varphi\left(e_i, e_j\right) = \left(\beta_p\left(e_i, e_j\right)\right)$$

At last we have

$$d^2\varphi = \beta \tag{9}$$

5. Charts

Let $n \in \mathbb{N}_+$.
Let M be a smooth manifold of dimension n.
Let $x = \mathrm{id}_{\mathbb{R}^m}$.
Let u be a chart of M.
Then we have:

$\mathrm{G}u$ is a smooth open submanifold of M

$u(\mathrm{G}u)$ is a smooth open submanifold of $\mathbb{R}^n$

$\mathrm{T}(\mathrm{G}u)$ is a smooth open submanifold of $\mathrm{T}M$

$\mathrm{T}\left(u(\mathrm{G}u)\right)$ is a smooth open submanifold of $\mathrm{T}\left(\mathbb{R}^n\right)$

$$u_\star \left(u^{-1}\right)_\star = \left(\mathrm{id}_{u(\mathrm{G}u)}\right)_\star = \mathrm{id}_{\mathrm{T}(u(\mathrm{G}u))}$$

$$\left(u^{-1}\right)_\star u_\star = \left(\mathrm{id}_{\mathrm{G}u}\right)_\star = \mathrm{id}_{\mathrm{T}(\mathrm{G}u)}$$

$u_\star : \mathrm{T}(\mathrm{G}u) \to \mathrm{T}\left(u(\mathrm{G}u)\right)$ is a diffeomorphism

$$\left(u_\star\right)^{-1} = \left(u^{-1}\right)_\star$$

$$\forall k \in \{1, \dots, n\} \quad \frac{\partial}{\partial u_k} = \left(u^{-1}\right)_\star \left(\frac{\partial}{\partial x_k} \circ u\right)$$

$$\forall k \in \{1, \dots, n\} \quad u_\star \frac{\partial}{\partial u_k} = \frac{\partial}{\partial x_k} \circ u$$

6. Application of 4

Theorem:

Pre.: Let $n \in \mathbb{N}_+$ be with $n \geq 2$.

Let $(M, <\dots;\dots>)$ be a Riemannian C^∞-manifold of dimension n.
Let ∇ a affine connection of M.
Let ∇ be compatible with the metric $<\dots;\dots>$.
Let T the Torsion tensor field of ∇.
Let u be a chart of M with $\mathrm{G}u = M$.
Let $u(\mathrm{G}u) = u(M) \subseteq \mathbb{R}^n$ be star-shaped.
We define the so called "Christoffel-Symbols" through

$$\forall i, j, k \in \{1, \dots, n\} \quad \Gamma^k_{ij} := \underbrace{< \nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} ; \frac{\partial}{\partial u_k} >}_{\in C^\infty(M,\mathbb{R})}$$

Let $(e_1, \dots, e_n)$ be the canonical base of $\mathbb{R}^n$.
We define a C^∞-mapping $g : u(\mathrm{G}u) \to \mathfrak{L}^2(\mathbb{R}^n, \mathbb{R})$ through

$$\forall q \in u(\mathrm{G}u) \quad \forall a, b \in \mathbb{R}^n \quad g_q(a, b) :=$$

$$= \sum_{j,k=1}^{n} a_j b_k \left(< \left(u^{-1}\right)_* \frac{\partial}{\partial x_j} ; \left(u^{-1}\right)_* \frac{\partial}{\partial x_k} > \right)_q$$

Ass.: Let for all $\forall k \in \{1, \dots, n\}$ $\varphi_k \in C^2(u(\mathrm{G}u), \mathbb{R})$ an anti-derivative of $g(e_k, \dots) : u(\mathrm{G}u) \to \mathfrak{L}(\mathbb{R}^n, \mathbb{R})$. Then the following is true:

$$\mathrm{T} = 0 \quad \Leftrightarrow$$

$$\left(\begin{array}{l} \forall k \in \{1, \dots, n\} \quad \forall i, j \in \{1, \dots, n\} \\ \left(\Gamma^k_{ij} \circ u^{-1}\right) + \frac{1}{2} \mathrm{d}\left(g\left(e_i, e_j\right)\right)\left(e_k\right) = \mathrm{d}^2\left(\varphi_k\right)\left(e_i, e_j\right) \end{array} \right)$$

Rem.: We apply 4. and get for all $k \in \{1, \dots, n\}$:

$$\begin{pmatrix} \text{There exists an antiderivative of} \\ g\left(e_k, \dots\right) : u\left(Gu\right) \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right) \end{pmatrix} \Leftrightarrow$$
$$\begin{pmatrix} \forall i, j \in \{1, \dots, n\} \\ \quad d\left(g\left(e_k, e_j\right)\right)\left(e_i\right) = d\left(g\left(e_k, e_i\right)\right)\left(e_j\right) \end{pmatrix}$$

Proof: The case "$\Leftarrow$" is trivial. So let be T = 0.

Let $k \in \{1, \dots, n\}$. Because of T = 0 and because ∇ is compatible with the metric $< \dots ; \dots >$, there is a known formula for the "Christoffel-Symbols". For every $i, j \in \{1, \dots, n\}$ it holds:

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial u_i} \bullet < \frac{\partial}{\partial u_j} ; \frac{\partial}{\partial u_k} > + \right.$$
$$+ \frac{\partial}{\partial u_j} \bullet < \frac{\partial}{\partial u_k} ; \frac{\partial}{\partial u_i} > +$$
$$\left. - \frac{\partial}{\partial u_k} \bullet < \frac{\partial}{\partial u_i} ; \frac{\partial}{\partial u_j} > \right)$$

The following is elementary:

$$\forall \iota \in \{1, \dots, n\} \qquad \frac{\partial}{\partial u_\iota} = \left(u^{-1}\right)_\star \frac{\partial}{\partial x_\iota} \circ u$$

With this we have for every $\iota, \kappa, \lambda \in \{1, \dots, n\}$:

$$\left(\frac{\partial}{\partial u_\iota} \bullet < \frac{\partial}{\partial u_\kappa} ; \frac{\partial}{\partial u_\lambda} >\right) \circ u^{-1} =$$

$$= \left(\left(\left(u^{-1}\right)_\star \frac{\partial}{\partial x_\iota} \circ u\right) \bullet < \frac{\partial}{\partial u_\kappa} ; \frac{\partial}{\partial u_\lambda} >\right) \circ u^{-1} =$$

$$= \left(\left(u^{-1}\right)_\star \frac{\partial}{\partial x_\iota}\right) \bullet < \frac{\partial}{\partial u_\kappa} ; \frac{\partial}{\partial u_\lambda} > \quad =$$

$$= \frac{\partial}{\partial x_\iota} \bullet \left(< \frac{\partial}{\partial u_\kappa} ; \frac{\partial}{\partial u_\lambda} > \circ \; u^{-1} \right) =$$

$$= \frac{\partial}{\partial x_\iota} \bullet \left(< \left(u^{-1}\right)_\star \frac{\partial}{\partial x_\kappa} ; \left(u^{-1}\right)_\star \frac{\partial}{\partial x_\lambda} > \right)$$

We remember the mapping $g : u(\mathrm{G}u) \to \mathfrak{L}^2\left(\mathbb{R}^n, \mathbb{R}\right)$. Then we have for all $i, j \in \{1, \dots, n\}$:

$$\Gamma^k_{ij} \circ u^{-1} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} \bullet g\left(e_j, e_k\right) + \frac{\partial}{\partial x_j} \bullet g\left(e_k, e_i\right) + \right.$$

$$\left. - \frac{\partial}{\partial x_k} \bullet g\left(e_i, e_j\right)\right) =$$

$$= \frac{1}{2}\left(\left(\mathrm{d}\left(g\left(e_k, e_j\right)\right)\left(e_i\right)\right) + \left(\mathrm{d}\left(g\left(e_k, e_i\right)\right)\left(e_j\right)\right) + \right.$$

$$\left. - \left(\mathrm{d}\left(g\left(e_i, e_j\right)\right)\left(e_k\right)\right)\right)$$

respectively

$$\left(\Gamma^k_{ij} \circ u^{-1}\right) + \frac{1}{2}\,\mathrm{d}\left(g\left(e_i, e_j\right)\right)\left(e_k\right) =$$

$$= \frac{1}{2}\left(\mathrm{d}\left(g\left(e_k, e_j\right)\right)\left(e_i\right) + \mathrm{d}\left(g\left(e_k, e_i\right)\right)\left(e_j\right)\right) =$$

$$= \mathrm{d}\left(g\left(e_k, e_j\right)\right)\left(e_i\right) =$$

$$= \mathrm{d}^2\left(\varphi_k\right)\left(e_i, e_j\right)$$

7. Literature

[1] http://WWW.Reinbothe.de/english/download/Geometry.pdf

[2] http://WWW.Reinbothe.de/english/download/Tensors.pdf