1. Notation

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.

1. We define $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ as the $\mathbb{R}$-vectorspace of all r-times $\mathbb{R}$-multilinear mappings $f : \underbrace{\mathbb{R}^m \times \ldots \times \mathbb{R}^m}_{r\text{-times}} \to \mathbb{R}$.

2. We define S_r as the group of all permutations $\sigma : \{1, \ldots, r\} \to \{1, \ldots, r\}$. Further we define $\mathrm{sgn}_r(\ldots)$ as the signum-function of S_r.

3. Let $f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$. We define:

$$\left(f \text{ is alternating}\right) \quad :\Leftrightarrow \left(\begin{array}{c} \forall \pi \in \mathrm{S}_r \quad \forall v_1, \ldots, v_r \in \mathbb{R}^m \quad f\left(v_1, \ldots, v_r\right) = \\ \mathrm{sgn}_r(\pi)\, f\left(v_{\pi(1)}, \ldots, v_{\pi(r)}\right) \end{array}\right)$$

4. We define:

$$\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) := \left\{f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) : \ f \text{ ist alternating}\right\}$$

$\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is a $\mathbb{R}$-subvectorspace of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

5. Let $f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$. We define:

$$\left(f \text{ ist symmetrical}\right) \quad :\Leftrightarrow$$
$$\left(\forall \pi \in S_r \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad f\left(v_1, \dots, v_r\right) = f\left(v_{\pi(1)}, \dots, v_{\pi(r)}\right)\right)$$

6. We define:

$$\mathfrak{L}^r_{\text{sym}}\left(\mathbb{R}^m, \mathbb{R}\right) := \left\{f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) : \quad f \text{ ist symmetrical}\right\}$$

$\mathfrak{L}^r_{\text{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is a $\mathbb{R}$ -subvectorspace of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

7. Let V be a finite dimensional $\mathbb{R}$ -vectorspace. Let $f : V \to V$ a $\mathbb{R}$ -linear mapping. Let U be a $\mathbb{R}$ -subvectorspace of V. We define:

$$f \text{ is a projection} \quad :\Leftrightarrow \quad f \circ f = f$$

and

$$\left(f \text{ is a projection of } V \text{ onto } U\right) \quad :\Leftrightarrow$$
$$\left(f \circ f = f \quad \text{and} \quad \operatorname{img}(f) = U\right)$$

Let f be a projection of V onto U. Than we have:

$$\left(\forall u \in U \quad f(u) = u\right) \quad \text{and} \quad \left(\operatorname{kern}(f)\right) \cap U = \{0\}$$

8. Abbreviations:

„Theo." means „Theorem".
„Pre." means „Premises".
„Ass." means „Assertion".
„i.e." means „that is".

2. Alternating Multilinear Forms

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.

Now we define a $\mathbb{R}$-linear mapping $\mathrm{pr}_{\mathrm{alt}}^{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}_{\mathrm{alt}}^{r,m}(\varphi)\right)\left(v_1, \dots, v_r\right) :=$$
$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\ \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

Then we have the following theorem:

Theo.:

Ass.: $\mathrm{pr}_{\mathrm{alt}}^{r,m}$ is a projection of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ onto $\mathfrak{L}_{\mathrm{alt}}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

Theo.:

Pre.: Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.
Let G be an open subset of $\mathbb{R}^m$.
Let $\omega : G \to \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$ be continuous differentiable.
Let $p \in G$.
We define $\zeta \in \mathfrak{L}^{r+1}\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall w_0, \dots, w_r \in \mathbb{R}^m \quad \zeta\left(w_0, \dots, w_r\right) :=$$
$$:= \left(\mathrm{d}_p \left(\omega_{\dots}\left(w_0, \dots, w_{r-1}\right)\right)\right)\left(w_r\right)$$

Ass.:
$$\forall w_0, \dots, w_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r+1,m}_{\mathrm{alt}}(\zeta)\right)\left(w_0, \dots, w_r\right) =$$
$$= \frac{(-1)^r}{r+1} \sum_{k=0}^{r} (-1)^k \left(\mathrm{d}_p \left(\omega_{\dots}\left(w_0, \dots, w_{k-1}, w_{k+1}, \dots, w_r\right)\right)\right)\left(w_k\right)$$

3. Symmetrical Multilinear Forms

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.

Now we define a $\mathbb{R}$-linear mapping $\mathrm{pr}^{r,m}_{\mathrm{sym}} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \ldots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r,m}_{\mathrm{sym}}(\varphi)\right)\left(v_1, \ldots, v_r\right) :=$$

$$:= \frac{1}{r!} \sum_{\sigma \in S_r} \varphi\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right)$$

Sei $k \in \{1, \ldots, r+1\}$.
We now define $\lambda_{k,r} \in S_{r+1}$ by

$$\forall i \in \{1, \ldots, r+1\} \quad \lambda_{k,r}(i) := \begin{cases} i & i < k \wedge i < r+1 \\ i+1 & i \geq k \wedge i < r+1 \\ k & i = r+1 \end{cases}$$

One can describe $\lambda_{k,r}$ by

$$\lambda_{k,r} = \begin{pmatrix} 1 & \ldots & k-1 & k & \ldots & r & r+1 \\ 1 & \ldots & k-1 & k+1 & \ldots & r+1 & k \end{pmatrix} \qquad (*)$$

Theo.:

Ass.: $\mathrm{pr}^{r,m}_{\mathrm{sym}}$ is a projection of $\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ onto $\mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$.

Bew.: In 3 steps:

1. $\mathrm{pr}^{r,m}_{\mathrm{sym}}\left(\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)\right) \subseteq \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is true, i.e. we have to prove:

$$\forall f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \mathrm{pr}^{r,m}_{\mathrm{sym}}(f) \text{ ist symmetrical} \tag{1}$$

Proof of (1):

Let $f \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$.

Let $v_1, \ldots, v_r \in \mathbb{R}^m$.

Let $\pi \in S_r$.

We now define $w_1, \ldots, w_r \in \mathbb{R}^m$ by

$$\forall i \in \{1, \ldots, r\} \quad w_i := v_{\pi(i)} \tag{2}$$

Then we have:

$$\forall \kappa \in S_r \quad \forall j \in \{1, \ldots, r\} \quad w_{\kappa(j)} = v_{\pi\circ\kappa(j)} \tag{3}$$

Now the following is true:

$$\left(\mathrm{pr}^{r,m}_{\mathrm{sym}}(f)\right)\left(v_1, \ldots, v_r\right) =$$

$$= \frac{1}{r!} \sum_{\sigma\in S_r} f\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right) =$$

$$= \frac{1}{r!} \sum_{\sigma\in S_r} f\left(v_{\pi\circ\sigma(1)}, \ldots, v_{\pi\circ\sigma(r)}\right)$$

With (2) and (3) we have:

$$\left(\mathrm{pr}_{\mathrm{sym}}^{r,m}(f)\right)\left(v_1,\dots,v_r\right) =$$

$$= \frac{1}{r!}\sum_{\sigma\in S_r} f\left(w_{\sigma(1)},\dots,w_{\sigma(r)}\right) =$$

$$= \left(\mathrm{pr}_{\mathrm{sym}}^{r,m}(f)\right)\left(w_1,\dots,w_r\right) =$$

$$= \left(\mathrm{pr}_{\mathrm{sym}}^{r,m}(f)\right)\left(v_{\pi(1)},\dots,v_{\pi(r)}\right)$$

2. $\mathrm{pr}^{r,m}_{\mathrm{sym}}\left(\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)\right) \supseteq \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$ is true, i.e. because of $\mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right) \subseteq \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ we have to prove:

$$\forall g \in \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right) \quad \mathrm{pr}^{r,m}_{\mathrm{sym}}(g) = g \tag{4}$$

Proof of (4):

Let $g \in \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$.

Let $v_1, \dots, v_r \in \mathbb{R}^m$.
Then we have:

$$\begin{aligned}&\left(\mathrm{pr}^{r,m}_{\mathrm{sym}}(g)\right)\left(v_1, \dots, v_r\right) = \\ &\quad = \frac{1}{r\,!} \sum_{\sigma \in S_r} g\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)\end{aligned} \tag{5}$$

Because $g \in \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$ the following is true:

$$\begin{aligned}\forall \sigma \in S_r \quad & g\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right) = \\ &= g\left(v_1, \dots, v_r\right)\end{aligned} \tag{6}$$

Moreover the following is true:

$$\# S_r = r\,! \tag{7}$$

With (5) - (7) we have:

$$\left(\mathrm{pr}^{r,m}_{\mathrm{sym}}(g)\right)\left(v_1, \dots, v_r\right) = g\left(v_1, \dots, v_r\right)$$

3. $\mathrm{pr}^{r,m}_{\mathrm{sym}} \circ \mathrm{pr}^{r,m}_{\mathrm{sym}} = \mathrm{pr}^{r,m}_{\mathrm{sym}}$ is true, i.e. we have to prove:

$$\forall h \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \left(\mathrm{pr}^{r,m}_{\mathrm{sym}} \circ \mathrm{pr}^{r,m}_{\mathrm{sym}}\right)(h) = \mathrm{pr}^{r,m}_{\mathrm{sym}}(h)$$

But this is a consequence of (1) and (4).

Theo.:

Pre.: Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$.
Let G be an open subset of $\mathbb{R}^m$.
Let $\omega : G \to \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$ be continuous differentiable.
Let $p \in G$.
We define $\zeta \in \mathfrak{L}^{r+1}\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\forall w_0, \dots, w_r \in \mathbb{R}^m \quad \zeta\left(w_0, \dots, w_r\right) := \\ := \left(\mathrm{d}_p\left(\omega_{\dots}\left(w_0, \dots, w_{r-1}\right)\right)\right)\left(w_r\right)$$

Ass.:

$$\forall w_0, \dots, w_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r+1,m}_{\mathrm{sym}}(\zeta)\right)\left(w_0, \dots, w_r\right) = \\ = \frac{1}{r+1}\sum_{k=0}^{r}\left(\mathrm{d}_p\left(\omega_{\dots}\left(w_0, \dots, w_{k-1}, w_{k+1}, \dots, w_r\right)\right)\right)\left(w_k\right)$$

Bew.: Let $v_1, \dots, v_{r+1} \in \mathbb{R}^m$.

We have per definitionem:

$$\left(\mathrm{pr}^{r+1,m}_{\mathrm{sym}}(\zeta)\right)\left(v_1, \dots, v_{r+1}\right) = \\ = \frac{1}{(r+1)!}\sum_{\sigma \in S_{r+1}} \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) \tag{1}$$

Now the following is true:

$$\sum_{\sigma\in S_{r+1}} \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) =$$

$$= \sum_{k=1}^{r+1} \sum_{\substack{\sigma\in S_{r+1} \\ \sigma(r+1)=k}} \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}, v_k\right) \tag{2}$$

and (proof of (3) later)

$$\forall k \in \{1, \dots, r+1\} \quad \forall \begin{pmatrix} \sigma \in S_{r+1} \\ \sigma(r+1) = k \end{pmatrix}$$

$$\zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) = \tag{3}$$

$$= \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right)$$

and

$$\forall k \in \{1, \dots, r+1\} \quad \#\left\{\sigma \in S_{r+1} \;:\; \sigma(r+1) = k\right\} = r! \tag{4}$$

With (1) - (4) the following is true:

$$\left(\mathrm{pr}_{\mathrm{sym}}^{r+1,m}(\zeta)\right)\left(v_1, \dots, v_{r+1}\right) =$$

$$= \frac{r!}{(r+1)!} \sum_{k=1}^{r+1} \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right) =$$

$$= \frac{1}{r+1} \sum_{k=1}^{r+1} \left(\mathrm{d}_p\left(\omega_{\dots}\left(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_{r+1}\right)\right)\right)\left(v_k\right)$$

Now follows the proof of (3):

Let $k \in \{1, \dots, r+1\}$.
Let $\sigma \in S_{r+1}$ and be $\sigma(r+1) = k$ true.
We now define $\pi \in S_{r+1}$ by

$$\pi := \sigma^{-1} \circ \lambda_{k,r} \in S_{r+1} \tag{5}$$

and $u_1, \dots, u_r \in \mathbb{R}^m$ by

$$\forall i \in \{1, \dots, r\} \quad u_i := v_{\sigma(i)} \tag{6}$$

Because $\sigma(r+1) = k$, the following is true:

$$\pi(r+1) = r+1 \tag{7}$$

Especially we have:

$$\forall j \in \{1, \dots, r\} \quad \left(\pi(\mathrm{j}) \in \{1, \dots, r\} \text{ und } u_{\pi(j)} = v_{\sigma\circ\pi(j)}\right) \tag{8}$$

Finally we get with (5) - (8) and $\forall q \in G \quad \omega_q \in \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$:

$$\begin{aligned}
&\zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r+1)}\right) = \\
&\quad = \zeta\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}, v_k\right) = \\
&\quad = \zeta\left(u_1, \dots, u_r, v_k\right) = \\
&\quad = \zeta\left(u_{\pi(1)}, \dots, u_{\pi(r)}, v_k\right) = \\
&\quad = \zeta\left(v_{\sigma\circ\pi(1)}, \dots, v_{\sigma\circ\pi(r)}, v_k\right) = \\
&\quad = \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r)}, v_k\right) = \\
&\quad = \zeta\left(v_{\lambda_{k,r}(1)}, \dots, v_{\lambda_{k,r}(r+1)}\right)
\end{aligned}$$

4. Projections

Theo.:

Pre.: Let V be a finite-dimensional $\mathbb{R}$ −vectorspace.
Let $f : V \to V$ be a projection.
Let $g : V \to V$ defined by $g := \mathrm{id}_V - f$.

Ass.: $g : V \to V$ is a projection with $f \circ g = g \circ f = 0$

Proof: We have to show:

$$g \circ g = g \tag{1}$$

and

$$f \circ g = g \circ f = 0 \tag{2}$$

Proof of (1):

$$\begin{aligned} g \circ g &= \left(\mathrm{id}_V - f\right) \circ \left(\mathrm{id}_V - f\right) = \\ &= \left(\mathrm{id}_V \circ \left(\mathrm{id}_V - f\right)\right) - \left(f \circ \left(\mathrm{id}_V - f\right)\right) = \\ &= \left(\mathrm{id}_V \circ \mathrm{id}_V\right) - \left(\mathrm{id}_V \circ f\right) - \left(f \circ \mathrm{id}_V\right) + \left(f \circ f\right) = \\ &= \mathrm{id}_V - f - f + \left(f \circ f\right) = \\ &= \mathrm{id}_V - f = \\ &= g \end{aligned}$$

Proof of (2):

$$\begin{aligned} f \circ g &= f \circ \left(\mathrm{id}_V - f\right) = \\ &= \left(f \circ \mathrm{id}_V\right) - \left(f \circ f\right) = \\ &= f - \left(f \circ f\right) \\ &= 0 \end{aligned}$$

and

$$
\begin{aligned}
g \circ f &= \left(\mathrm{id}_V - f\right) \circ f = \\
&= \left(\mathrm{id}_V \circ f\right) - \left(f \circ f\right) = \\
&= f - \left(f \circ f\right) \\
&= 0
\end{aligned}
$$

Satz:

Vor.: Let V be a finite-dimensional $\mathbb{R}$ -vectorspace.
Let $k \in \mathbb{N}_+$.
Let for every $i \in \{1, \dots, k\}$ $P_i : V \to V$ be a projection.

Beh.: $\left(\forall i, j \in \{1, \dots, k\} \quad \left(i \neq j \quad \Rightarrow \quad P_i \circ P_j = 0\right)\right) \quad \Leftrightarrow$

$$\left(\begin{array}{l} \sum_{i=1}^{k} P_i : V \to V \text{ is a projection with} \\ \operatorname{img}\left(\sum_{i=1}^{k} P_i\right) = \bigoplus_{i=1}^{k} \operatorname{img}\left(P_i\right) \end{array}\right)$$

Bew.:

„=>“: We assume $\left(\forall i, j \in \{1, \dots, k\} \quad \left(i \neq j \quad \Rightarrow \quad P_i \circ P_j = 0\right)\right)$.

Then we have:

$$\begin{aligned} \left(\sum_{i=1}^{k} P_i\right) \circ \left(\sum_{j=1}^{k} P_j\right) &= \sum_{i=1}^{k} \left(P_i \circ \left(\sum_{j=1}^{k} P_j\right)\right) = \\ &= \sum_{i=1}^{k} \sum_{j=1}^{k} \left(P_i \circ P_j\right) = \\ &= \sum_{i=1}^{k} \left(P_i \circ P_i\right) = \\ &= \sum_{i=1}^{k} P_i \end{aligned} \tag{1}$$

i.e.

$$\sum_{i=1}^{k} P_i : V \to V \text{ is a projection} \tag{2}$$

Obviously the following is true:

$$\operatorname{img}\left(\sum_{i=1}^{k} P_i\right) \subseteq \sum_{i=1}^{k} \operatorname{img}\left(P_i\right) \tag{3}$$

With the assumption we have:

$$\left(\forall i, j \in \{1, \dots, k\} \quad \left(i \neq j \quad \Rightarrow \quad \operatorname{img}\left(P_j\right) \subseteq \ker\left(P_i\right)\right)\right) \tag{4}$$

Because for every $i \in \{1, \dots, k\}$ $P_i : V \to V$ is a projection, the following is true:

$$\left(\forall i \in \{1, \dots, k\} \quad \ker\left(P_i\right) \cap \operatorname{img}\left(P_i\right) = \{0\}\right)$$

Then we have:

$$\left(\forall i, j \in \{1, \dots, k\} \quad \left(i \neq j \quad \Rightarrow \quad \operatorname{img}\left(P_i\right) \cap \operatorname{img}\left(P_j\right) = \{0\}\right)\right)$$

With that it is shown:

$$\operatorname{img}\left(\sum_{i=1}^{k} P_i\right) \subseteq \bigoplus_{i=1}^{k} \operatorname{img}\left(P_i\right) \tag{5}$$

Then it remains to show:

$$\forall j \in \{1, \dots, k\} \quad \operatorname{img}\left(P_j\right) \subseteq \operatorname{img}\left(\sum_{i=1}^{k} P_i\right) \tag{6}$$

Proof of this:

Let $j \in \{1, \dots, k\}$.
With (4) we have finally:

$$\operatorname{img}\left(\sum_{i=1}^{k} P_i\right) = \left(\sum_{i=1}^{k} P_i\right)(V) \supseteq \left(\sum_{i=1}^{k} P_i\right)\left(\operatorname{img}\left(P_j\right)\right) =$$

$$= P_j\left(\operatorname{img}\left(P_j\right)\right) = \operatorname{img}\left(P_j \circ P_j\right) = \operatorname{img}\left(P_j\right)$$

„<=“: We now assume:

$$\left(\begin{array}{l} \sum_{j=1}^{k} P_j : \ V \to \ V \ \text{ is a projection with} \\ \operatorname{img}\left(\sum_{j=1}^{k} P_j \right) = \bigoplus_{j=1}^{k} \operatorname{img}\left(P_j\right) \end{array} \right)$$

The following has to be shown:

$$\forall i, j \in \{1, \dots, k\} \qquad \left(i \neq j \quad \Rightarrow \quad \operatorname{img}\left(P_i\right) \subseteq \ker\left(P_j\right)\right)$$

Let $i \in \{1, \dots, k\}$.

Let $u \in \operatorname{img}\left(P_i\right)$.

Because $\sum_{j=1}^{k} P_j$ and P_i are projections and because $u \in \operatorname{img}\left(\sum_{j=1}^{k} P_j \right) = \bigoplus_{j=1}^{k} \operatorname{img}\left(P_j\right)$, we have:

$$u = \sum_{j=1}^{k} P_j (u) = u + \sum_{\substack{j=1 \\ i \neq j}}^{k} P_j (u)$$

and now

$$\sum_{\substack{j=1 \\ i \neq j}}^{k} P_j (u) = 0$$

Because $\operatorname{img}\left(\sum_{j=1}^{k} P_j \right) = \bigoplus_{j=1}^{k} \operatorname{img}\left(P_j\right)$, we have finally:

$$\forall j \in \{1, \dots, k\} \qquad \left(i \neq j \quad \Rightarrow \quad \left(P_j (u) = 0\right)\right)$$

5. Projections and Tensors

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$ be with $r \geq 2$.

We now define a $\mathbb{R}$-linear mapping $\mathrm{pr}_1^{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\mathrm{pr}_1^{r,m} := \mathrm{pr}_{\mathrm{alt}}^{r,m} + \mathrm{pr}_{\mathrm{sym}}^{r,m}$$

Then the following is true:

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \ldots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}_1^{r,m}(\varphi)\right)\left(v_1, \ldots, v_r\right) := \\ := \frac{2}{r!} \sum_{\substack{\sigma \in S_r \\ \mathrm{sgn}_r(\sigma) = 1}} \varphi\left(v_{\sigma(1)}, \ldots, v_{\sigma(r)}\right)$$

Theo:

Ass.: $\mathrm{pr}_1^{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ is a projection onto $\underbrace{\mathfrak{L}_{\mathrm{alt}}^r\left(\mathbb{R}^m, \mathbb{R}\right)}_{=\mathrm{img}\left(\mathrm{pr}_{\mathrm{alt}}^{r,m}\right)} \oplus \underbrace{\mathfrak{L}_{\mathrm{sym}}^r\left(\mathbb{R}^m, \mathbb{R}\right)}_{=\mathrm{img}\left(\mathrm{pr}_{\mathrm{sym}}^{r,m}\right)}$

Proof: We only have to show:

$$\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) \cap \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right) = \{0\}$$

Proof of this:

Let $f \in \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right) \cap \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$.

Let $v_1, \dots, v_r \in \mathbb{R}^m$.

Let $\tau \in S_r$ be defined by (Cave! $r \geq 2$):

$$\begin{aligned} \tau : \{1, \dots, r\} &\to \{1, \dots, r\} \\ i &\mapsto \tau(i) := \begin{cases} 2 & i = 1 \\ 1 & i = 2 \\ i & i \geq 3 \end{cases} \end{aligned}$$

Than we have:

$$\mathrm{sgn}_r(\tau) = -1$$

Because $r \geq 2$ and $f \in \mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)$, the following is true:

$$f\left(v_1, \dots, v_r\right) = \mathrm{sgn}_r(\tau)\, f\left(v_{\tau(1)}, \dots, v_{\tau(r)}\right)$$

Because $r \geq 2$ and $f \in \mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)$, the following is true:

$$f\left(v_1, \dots, v_r\right) = f\left(v_{\tau(1)}, \dots, v_{\tau(r)}\right)$$

Then we have:

$$f\left(v_1, \dots, v_r\right) = 0$$

We now define a $\mathbb{R}$-linear mapping $\mathrm{pr}_0^{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ by

$$\mathrm{pr}_0^{r,m} \coloneqq \mathrm{id}_{\mathfrak{L}^r\left(\mathbb{R}^m,\mathbb{R}\right)} - \mathrm{pr}_1^{r,m}$$

Theo:

Ass.: $\mathrm{pr}_0^{r,m} : \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \to \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right)$ is a projection with $\mathrm{pr}_0^{r,m} \circ \mathrm{pr}_1^{r,m} = \mathrm{pr}_1^{r,m} \circ \mathrm{pr}_0^{r,m} = 0$

Proof: Clear with theorems in 4. and 5.

6. Result

Let $m \in \mathbb{N}_+$.
Let $r \in \mathbb{N}_+$ be with $r \geq 2$.

The following is true:

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r,m}_{\mathrm{alt}}(\varphi)\right)\left(v_1, \dots, v_r\right) := $$
$$= \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}_r(\sigma)\ \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

and

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r,m}_{\mathrm{sym}}(\varphi)\right)\left(v_1, \dots, v_r\right) := $$
$$= \frac{1}{r!} \sum_{\sigma \in S_r} \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

and

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r,m}_{1}(\varphi)\right)\left(v_1, \dots, v_r\right) := $$
$$= \frac{2}{r!} \sum_{\substack{\sigma \in S_r \\ \mathrm{sgn}_r(\sigma) = 1}} \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

and

$$\forall \varphi \in \mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) \quad \forall v_1, \dots, v_r \in \mathbb{R}^m \quad \left(\mathrm{pr}^{r,m}_{0}(\varphi)\right)\left(v_1, \dots, v_r\right) := $$
$$= \varphi\left(v_1, \dots, v_r\right) - \frac{2}{r!} \sum_{\substack{\sigma \in S_r \\ \mathrm{sgn}_r(\sigma) = 1}} \varphi\left(v_{\sigma(1)}, \dots, v_{\sigma(r)}\right)$$

Finally the following is true:

$$\begin{pmatrix} \mathrm{pr}_0^{r,m} \text{ and } \mathrm{pr}_1^{r,m} \text{ are projections with} \\ \mathrm{pr}_0^{r,m} \circ \mathrm{pr}_1^{r,m} = \mathrm{pr}_1^{r,m} \circ \mathrm{pr}_0^{r,m} = 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{pr}_0^{r,m}, \mathrm{pr}_{\mathrm{alt}}^{r,m} \text{ and } \mathrm{pr}_{\mathrm{sym}}^{r,m} \text{ are projections with} \\ \mathrm{pr}_{\mathrm{alt}}^{r,m} \circ \mathrm{pr}_0^{r,m} = \mathrm{pr}_0^{r,m} \circ \mathrm{pr}_{\mathrm{alt}}^{r,m} = 0 \quad \text{and} \\ \mathrm{pr}_0^{r,m} \circ \mathrm{pr}_{\mathrm{sym}}^{r,m} = \mathrm{pr}_{\mathrm{sym}}^{r,m} \circ \mathrm{pr}_0^{r,m} = 0 \quad \text{and} \\ \mathrm{pr}_{\mathrm{alt}}^{r,m} \circ \mathrm{pr}_{\mathrm{sym}}^{r,m} = \mathrm{pr}_{\mathrm{sym}}^{r,m} \circ \mathrm{pr}_{\mathrm{alt}}^{r,m} = 0 \end{pmatrix}$$

and

$$r = 2 \quad \Rightarrow \quad \mathrm{pr}_0^{r,m} = 0$$

and

$$\mathrm{id}_{\mathfrak{L}^r\left(\mathbb{R}^m,\mathbb{R}\right)} = \mathrm{pr}_0^{r,m} + \mathrm{pr}_1^{r,m} = \mathrm{pr}_0^{r,m} + \left(\mathrm{pr}_{\mathrm{alt}}^{r,m} + \mathrm{pr}_{\mathrm{sym}}^{r,m}\right)$$

and

$$\mathfrak{L}^r\left(\mathbb{R}^m, \mathbb{R}\right) = \mathrm{img}\left(\mathrm{pr}_0^{r,m}\right) \oplus \underbrace{\mathfrak{L}^r_{\mathrm{alt}}\left(\mathbb{R}^m, \mathbb{R}\right)}_{=\mathrm{img}\left(\mathrm{pr}_{\mathrm{alt}}^{r,m}\right)} \oplus \underbrace{\mathfrak{L}^r_{\mathrm{sym}}\left(\mathbb{R}^m, \mathbb{R}\right)}_{=\mathrm{img}\left(\mathrm{pr}_{\mathrm{sym}}^{r,m}\right)}$$