Generalized Vector Product

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1. Definition $\mathcal{L}^k(V, W)$

Def. Let V and W be real vector spaces.
Let $k \in \mathbb{N}_+$.
We define a real vector space $\mathcal{L}^k(V, W)$ through

$$\mathcal{L}^k(V, W) := \left\{ f : V^k \to W \ \ k-\text{times multilinear} \right\}$$

2. Definition V^*

Def. Let V be a real vector space.
We define a real vector space V^* through

$$V^* := \mathcal{L}^1(V, \mathbb{R})$$

3. Isomorphism V To V^*

Theo.

Pre. Let V be a real vector space.
Let $< \ldots ; \ldots >: \ V \times V \to \mathbb{R}$ be an inner product on V.

Ass. The mapping

$$\begin{aligned} V &\to V^* \\ w &\mapsto \left(< w ; \ldots >: \ V \to \mathbb{R}\right) \end{aligned}$$

is an isomorphism.

4. Generalization of 3.

Theo.

Pre. Let V be a real vector space.
Let $< \dots ; \dots >: \ V \times V \to \mathbb{R}$ be an inner product on V.
Let $k \in \mathbb{N}_+$.

Ass. The mapping

$$\begin{array}{ccc} \mathcal{L}^k(V,V) & \to & \mathcal{L}^{k+1}(V,\mathbb{R}) \\ \Theta & \mapsto & \begin{pmatrix} V \times \dots \times V & \to \mathbb{R} \\ \left(X_1, \dots, X_{k+1}\right) & \mapsto < \Theta\left(X_1, \dots, X_k\right); X_{k+1} > \end{pmatrix} \end{array}$$

is an isomorphism.

The inversal:

Set $n := \dim(V)$ and let $n \in \mathbb{N}_+$.
Let $E_1, \dots, E_n \in V$ be an orthonormal basis of $(V, < \dots ; \dots >)$.
Let $\vartheta \in \mathcal{L}^{k+1}(V,\mathbb{R})$.
We define $\Theta \in \mathcal{L}^k(V,V)$ through

$$\forall X_1, \dots, X_n \in V \quad \Theta\left(X_1, \dots, X_n\right) := \sum_{i=1}^{n} \vartheta\left(X_1, \dots, X_n, E_i\right) \cdot E_i$$

For this Θ the following is true:

$$\begin{aligned} \forall X_1, \dots, X_{n+1} \in V \quad & < \Theta\left(X_1, \dots, X_n\right); X_{n+1} >= \\ & =< \sum_{i=1}^{n} \left(\vartheta\left(X_1, \dots, X_n, E_i\right) \cdot E_i\right); X_{n+1} >= \\ & = \vartheta\left(X_1, \dots, X_n, \sum_{i=1}^{n} < E_i ; X_{n+1} > \cdot E_i\right) = \\ & = \vartheta\left(X_1, \dots, X_{n+1}\right) \end{aligned}$$

Observation 1:

If $\vartheta \in \mathcal{L}^{k+1}(V, \mathbb{R})$ is alternating, then $\Theta \in \mathcal{L}^{k}(V, V)$ is alternating.

Observation 2:

Let $\chi : \mathcal{L}^{k}(V, V) \to \mathcal{L}^{k+1}(V, \mathbb{R})$ be that isomorphism.
Let $f \in \mathcal{L}^{1}(V, V)$.
Let $f^{t} \in \mathcal{L}^{1}(V, V)$ the adjoint to f in $(V, < \dots ; \dots >)$.
Then we have:

$$\begin{aligned}\forall \Theta \in \mathcal{L}^{k}(V, V) \quad \forall X_1, \dots, X_{n+1} \in \mathrm{V} \quad (\chi(f \circ \Theta))\left(X_1, \dots, X_{n+1}\right) &= \\ &=< f\left(\Theta\left(X_1, \dots, X_n\right)\right); X_{n+1} >= \\ &=< \Theta\left(X_1, \dots, X_n\right); f^{t}\left(X_{n+1}\right) >= \\ &= (\chi(\Theta))\left(X_1, \dots, X_n, f^{t}\left(X_{n+1}\right)\right)\end{aligned}$$

This behaviour of χ is called "contravariant" (in the $(n+1)$th place).

A $\tilde{\chi} : \mathcal{L}^{k}(V, V) \to \mathcal{L}^{k+1}(V, \mathbb{R})$ with

$$\begin{aligned}\forall \Theta \in \mathcal{L}^{k}(V, V) \quad \forall X_1, \dots, X_{n+1} \quad (\tilde{\chi}(f \circ \Theta))\left(X_1, \dots, X_{n+1}\right) &= \\ &= (\tilde{\chi}(\Theta))\left(X_1, \dots, X_n, f\left(X_{n+1}\right)\right)\end{aligned}$$

would be called "covariant" (in the $(n+1)$th place).

5. Generalized Vector Product On $\mathbb{R}^{n+1}$

Theo.

Pre. Let $n \in \mathbb{N}_+$.

Let $< \ldots ; \ldots >: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be the standard inner product on $\mathbb{R}^{n+1}$.

Ass. There exists exactly one mapping

$$\Theta : \underbrace{\mathbb{R}^{n+1} \times \ldots \times \mathbb{R}^{n+1}}_{n-\text{times}} \to \mathbb{R}^{n+1}$$

with

$$\forall X_1, \ldots, X_{n+1} \in \mathbb{R}^{n+1} \quad < \Theta\left(X_1, \ldots, X_n\right); X_{n+1} > = \det\left(X_1, \ldots, X_{n+1}\right)$$

Rem. For the proof you need 3. and the axiom of choice.

Rem. In literature you may find

$$\forall X_1, \ldots, X_n \in \mathbb{R}^{n+1} \quad \Theta\left(X_1, \ldots, X_n\right) = X_1 \wedge \ldots \wedge X_n$$

Rem. For $n = 2$ $\Theta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the well known vector product $\ldots \times \ldots$ on $\mathbb{R}^3$.

Rem. $\Theta : \underbrace{\mathbb{R}^{n+1} \times \ldots \times \mathbb{R}^{n+1}}_{n-\text{times}} \to \mathbb{R}^{n+1}$ is n − times multilinear and alternating.