1. What does “canonical” mean?

1.1. Definition as of [1]

Def. I: A concept amongst a number of concepts is defined as canonical, iff it has a special meaning and an especialy transparent figure.

1.2. Definition as of [3]

Def. II: canonical, best adjusted to a given situation or problem

1.3. Definition as of [4]

Def.: III canonical, in a natural way logically distinguished

2. Problems with the "canonical" Base of $\mathbb{R}^2$

2.1. Thesis

The base $\mathfrak{B} := \left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right) \in \left(\mathbb{R}^2\right)^2$ of $\mathbb{R}^2$ is in no way logically distinguished against the base $\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right) \in \left(\mathbb{R}^2\right)^2$ of $\mathbb{R}^2$. $\mathfrak{B}$ is **not** "**canonical**" but **arbitrary** in the sense of the definitions I, II and III.

2.2. An Objection?

But $\det\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right) = 1 \neq -1 = \det\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)$ is true?

2.3. Solution

The definition of $\det(\ldots)$ is also **not "canonical"**, but **arbitrary**. It is:

$$\det\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} = \sum_{\pi\in S(n)}\left(\operatorname{sgn}(\pi)\left(\prod_{i=1}^{n} a_{i,\pi(i)}\right)\right)$$

The **arbitrariness** in this definition is the direction, in which the matrix is read. It is also possible, to define another determinant $\widetilde{\det}(\ldots)$ as follows:

$$\widetilde{\det}\begin{pmatrix} a_{n,1} & \cdots & a_{n,n} \\ \vdots & \iddots & \vdots \\ a_{1,1} & \cdots & a_{1,n} \end{pmatrix} := \sum_{\pi\in S(n)}\left(\operatorname{sgn}(\pi)\left(\prod_{i=1}^{n} a_{i,\pi(i)}\right)\right)$$

With $\widetilde{\det}(\ldots)$ the following is true:

$$\widetilde{\det}\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right) = -1 \neq 1 = \widetilde{\det}\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right)$$

The Thesis 2.1. is confirmed.

2.4. A Try

It comes to mind, to define the term "“**canonical**” base of $\mathbb{R}^2$" as follows:

$$\forall v, w \in \mathbb{R}^2 \quad \left(\left((v, w) \text{ is the canonical base of } \mathbb{R}^2\right) \quad :\Leftrightarrow \quad \left(\left(v = \begin{pmatrix}1\\0\end{pmatrix}\right) \wedge \left(w = \begin{pmatrix}0\\1\end{pmatrix}\right)\right)\right)$$

But that is **arbitrary**, because you could define the term "“**canonical**” base of $\mathbb{R}^2$" another way:

$$\forall v, w \in \mathbb{R}^2 \quad \left(\left((v, w) \text{ is the canonical base of } \mathbb{R}^2\right) \quad :\Leftrightarrow \quad \left(\left(v = \begin{pmatrix}0\\1\end{pmatrix}\right) \wedge \left(w = \begin{pmatrix}1\\0\end{pmatrix}\right)\right)\right)$$

The only thing, that is possible, is to define the term "standard-base of $\mathbb{R}^2$":

$$\forall v, w \in \mathbb{R}^2 \quad \left(\left((v, w) \text{ is the standard-base of } \mathbb{R}^2\right) \quad :\Leftrightarrow \quad \left(\left(v = \begin{pmatrix}1\\0\end{pmatrix}\right) \wedge \left(w = \begin{pmatrix}0\\1\end{pmatrix}\right)\right)\right)$$

The "standard-base of $\mathbb{R}^2$" is **not** "**canonical**", but defined **arbitrary**. In other words:

The choice of the "standard-base of $\mathbb{R}^2$" is **favorably**, but **not mandatory**.

3.1. The Neutral Element of a Group G with $\#G \geq 2$ is Not "canonical"

Let G be a set with $\# G \geq 2$ and let $(G; \cdot)$ be a group with neutral element $e \in G$. It will be shown, that every $\vartheta \in G$ causes another group $(G; \odot)$, which is related with $(G; \cdot)$ and has neutral Element ϑ.

Let $\vartheta \in G$ and let $\odot : \; G \times G \to G$ be defined as

$$\forall a, b \in G \quad a \odot b := a \cdot \vartheta^{-1} \cdot b \qquad (*)$$

Let the mapping $\varphi : \; G \to G$ be defined as

$$\forall a \in G \quad \varphi(a) := \vartheta \cdot a^{-1} \cdot \vartheta \qquad (**)$$

With (*) and (**) we have:

$$\forall a, b, c \in G \quad a \odot (b \odot c) = (a \odot b) \odot c \qquad (1)$$

$$\forall a \in G \quad a \odot \vartheta = a = \vartheta \odot a \qquad (2)$$

$$\forall a \in G \quad a \odot \varphi(a) = \vartheta = \varphi(a) \odot a \qquad (3)$$

With (1) - (3) it is shown, that $(G; \odot)$ is a group with neutral element ϑ. Further $\varphi : \; G \to G$ is the inverse mapping of $(G; \odot)$. Now the relationship between $(G; \odot)$ and $(G; \cdot)$ is:

$$\forall a, b \in G \quad a \cdot b = a \odot \varphi(e) \odot b \qquad (4)$$

Result:

Because $\# G \geq 2$, the neutral element $e \in G$ of $(G; \cdot)$ is **not "canonical"**.

3.2. The Generating Element of a cyclic Group G with $\#G \geq 2$ is not "canonical"

Let G be a set with $\# G \geq 2$ and let $(G; \cdot)$ be a cyclic group with generating element $\omega \in G$. It will be shown, that every $g \in G$ causes another cyclic group $(G; \odot)$, which is related with $(G; \cdot)$ and has generating element g.

Let $g \in G$. Then there exists $k \in \{1, \dots, \# G\}$ with

$$g = \omega^k \qquad (1)$$

We define $l \in \{0, \dots, \# G - 1\}$ and $\vartheta \in G$ as follows:

$$l = k - 1 \quad \text{and} \quad \vartheta = \omega^l \qquad (2)$$

Let $\odot : G \times G \to G$ be defines as in 3.1.(*), respectively

$$\forall a, b \in G \quad a \odot b := a \cdot \vartheta^{-1} \cdot b \qquad (3)$$

Then we have with 3.1.(1) - 3.1.(3):

$$(G; \odot) \text{ is a group with neutral element } \vartheta \qquad (4)$$

With (1), (2) und (3) we have:

$$\forall m \in \{1, \dots, \# G\} \quad \bigodot_{i=1}^{m} g = g^m \; \vartheta^{-(m-1)} = \omega^{mk-(m-1)l} \tag{5}$$

Now is the question, wether $(G; \odot)$ is a cyclic group and wether $g \in G$ is a generating element of $(G; \odot)$. Because of (4) we have to show:

$$\forall n \in \{1, \dots, \# G\} \quad \left(\vartheta = \bigodot_{i=1}^{n} g \quad \Rightarrow \quad n = \# G \right) \tag{6}$$

Proof of (6):

Let $n \in \{1, \dots, \# G\}$ with $\vartheta = \bigodot_{i=1}^{n} g$. With (2) and (5) follows:

$$\omega^l = \vartheta = \omega^{nk-(n-1)l} = \omega^{nk-nl+l}$$

Let $e \in G$ be the neutral element of $(G; \cdot)$. With (2) follows:

$$e = \omega^l \; \omega^{\# G-l} = \omega^{nk-nl+l+\# G-l} = \omega^{nk-nl} = \omega^{n(k-l)} = \omega^n$$

Because $\omega \in G$ is a generating element of $(G; \cdot)$, we have at last:

$$n = \# G$$

Result:

Because 3.1.(4) and $\# G \geq 2$ the generating element $\omega \in G$ of $(G; \cdot)$ is **not "canonical"**.

4. Dual Spaces

4.1. Necessary definitions

Def.: Let $n \in \mathbb{N}_+$.
Let V be a n-dimensional $\mathbb{R}$-vector space.

1. We define

$$V^\star := \{f : V \to \mathbb{R} : \ f \text{ is } \mathbb{R}\text{-linear}\}$$

Then the following is true:

$$V^\star \text{ is a } n\text{-dimensional } \mathbb{R}\text{-vector space}$$

$V^\star$ is called the dual space of V.

2. Let $\|\ldots\|$ be a norm of V.
We then define a norm $\|\ldots\|_\star$ auf $V^\star$ through

$$\forall f \in V^\star \quad \|f\|_\star := \underbrace{\sup\left\{\frac{|f(x)|}{\|x\|} : \ x \in V \ \wedge \ x \neq 0\right\}}_{=\sup\{|f(x)|:\ x\in V \ \wedge \ \|x\|=1\}}$$

$\|\ldots\|_\star$ is called the on $V^\star$ inducted operator norm.

3. Let $<\ldots;\ldots>$ be a scalar product on V. We define a linear mapping $\Theta_{<\ldots;\ldots>,V} \;:\; V \to V^{\star}$ through

$$\forall x \in V \quad \Theta_{<\ldots;\ldots>,V}(x) :=< x;\ldots >$$

With $<\ldots;\ldots>$ we have a norm $\|\ldots\|$ on V:

$$\forall x \in V \quad \|x\| = \sqrt{< x; x >}$$

Then the following is true:

$$\Theta_{<\ldots;\ldots>,V} \;:\; (V, \|\ldots\|) \to \left(V^{\star}, \|\ldots\|_{\star}\right)$$
is an isometry of **normed** $\mathbb{R}$-vector spaces

4. Conducting from 1. and 2. $V^{\star\star} = \left(V^{\star}\right)^{\star}$ and $\|\ldots\|_{\star\star} = \left(\|\ldots\|_{\star}\right)_{\star}$ are also defined.

$V^{\star\star}$ is called double dual of V.

5. We define a mapping $Q_V \;:\; V \to V^{\star\star}$ through

$$\forall x \in V \;\; Q_V(x) := \underbrace{\begin{pmatrix} V^{\star} & \to & \mathbb{R} \\ f & \mapsto & f(x) \end{pmatrix}}_{\in V^{\star\star}}$$

With [2] we have:

$Q_V \;:\; V \to V^{\star\star}$ is $\mathbb{R}$-linear and bijektve

Furthermore [2] it holds true for every norm $\|\ldots\|$ of V:

$Q_V \;:\; (V, \|\ldots\|) \to \left(V^{\star\star}, \|\ldots\|_{\star\star}\right)$ is a $\mathbb{R}$-linear isometry of **normed** $\mathbb{R}$-vector spaces.

4.2. Theorem I

Theorem:

Pre.: Let $< \ldots ; \ldots >$ be a scalar product on $\mathbb{R}^2$.
Let $\|\ldots\|$ be the norm of $\mathbb{R}^2$, which is induced by $< \ldots ; \ldots >$.
Let $\Theta : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^*, \|\ldots\|_*\right)$ be a $\mathbb{R}$-linear isometry of normed $\mathbb{R}$-vector spaces.
Let $\Phi : \mathbb{R}^2 \to \left(\mathbb{R}^2\right)^*$ be a mapping.

Ass.:

$$\left(\begin{array}{l}\Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^*, \|\ldots\|_*\right) \text{ is a } \mathbb{R}\text{-linear} \\ \text{isometry of normed } \mathbb{R}\text{-vector spaces}\end{array}\right) \Leftrightarrow$$

$$\Phi \in \left\{ f : \mathbb{R}^2 \to \left(\mathbb{R}^2\right)^* : \left(\begin{array}{l}\text{There is a } \mathbb{R}\text{-linear} \\ \text{isometry} \\ g : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\mathbb{R}^2, \|\ldots\|\right) \\ \text{with } f = \Theta \circ g\end{array}\right)\right\}$$

Thus we have:

$$\left(\begin{array}{l}\Theta_{<\ldots;\ldots>,\mathbb{R}^2} : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{\star}, \|\ldots\|_{\star}\right) \text{ is a} \\ \mathbb{R}\text{-linear isometry of normed } \mathbb{R}\text{-vector spaces}\end{array}\right)$$

and

$$\left(\begin{array}{l}\left(-\Theta_{<\ldots;\ldots>,\mathbb{R}^2}\right) : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{\star}, \|\ldots\|_{\star}\right) \text{ is a} \\ \mathbb{R}\text{-linear isometry of normed } \mathbb{R}\text{-vector spaces}\end{array}\right)$$

and

$$\left(-\Theta_{<\ldots;\ldots>,\mathbb{R}^2}\right) = \left(\Theta_{<\ldots;\ldots>,\mathbb{R}^2}\right) \circ \left(-\mathrm{id}_{\mathbb{R}^2}\right)$$

and

$$\left(\Theta_{<\ldots;\ldots>,\mathbb{R}^2}\right) = \left(-\Theta_{<\ldots;\ldots>,\mathbb{R}^2}\right) \circ \left(-\mathrm{id}_{\mathbb{R}^2}\right)$$

and

$$\Theta_{<\ldots;\ldots>,\mathbb{R}^2} \neq -\Theta_{<\ldots;\ldots>,\mathbb{R}^2}$$

So it is clear:

$\Theta_{<\ldots;\ldots>,\mathbb{R}^2}$ is **not “canonical”**

4.3. Theorem II

Theorem:

Pre.: Let $\Phi : \mathbb{R}^2 \to \left(\mathbb{R}^2\right)^{**}$ be a mapping.

Ass.:

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ of } \mathbb{R}^2 \text{ is true:}\\ \Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{**}, \|\ldots\|_{**}\right) \text{ is a } \mathbb{R}\text{-linear}\\ \text{isometry of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \Leftrightarrow$$

$$\Phi \in \left\{Q_{\mathbb{R}^2}, -Q_{\mathbb{R}^2}\right\}$$

Proof: omitted

Thus we have:

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ of } \mathbb{R}^2 \text{ is true:}\\ Q_{\mathbb{R}^2} : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{**}, \|\ldots\|_{**}\right) \text{ is a } \mathbb{R}\text{-linear}\\ \text{isometry of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right)$$

and

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ of } \mathbb{R}^2 \text{ is true:}\\ \left(-Q_{\mathbb{R}^2}\right) : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{**}, \|\ldots\|_{**}\right) \text{ is a } \mathbb{R}\text{-linear}\\ \text{isometry of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right)$$

and

$$\left(-Q_{\mathbb{R}^2}\right) = \left(Q_{\mathbb{R}^2}\right) \circ \left(-\mathrm{id}_{\mathbb{R}^2}\right)$$

and

$$\left(Q_{\mathbb{R}^2}\right) = \left(-Q_{\mathbb{R}^2}\right) \circ \left(-\mathrm{id}_{\mathbb{R}^2}\right)$$

and

$$Q_{\mathbb{R}^2} \neq -Q_{\mathbb{R}^2}$$

So it is clear:

$Q_{\mathbb{R}^2}$ is **not “canonical”**

5. Index of Literature

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