Lemma: I

Pre.: Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{R}$-linear mapping.
Let <...;...> be a scalar product on $\mathbb{R}^2$.

Ass.: Φ is an isometry of $\left(\mathbb{R}^2, < \ldots ; \ldots >\right) \quad \Leftrightarrow$

$$\left(\begin{array}{l} \text{There is an orthonormal base of } \left(\mathbb{R}^2, < \ldots ; \ldots >\right), \\ \text{that the matrix of } \Phi \text{, which is related to this} \\ \text{base, has one of the following shapes:} \\ \text{a) } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \text{b) } \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix} \\ \quad \text{where } \vartheta \in [0; 2\pi[\end{array}\right)$$

Proof:

„$\Rightarrow$“: Let Φ be an isometry of $\left(\mathbb{R}^2, < \ldots ; \ldots >\right)$.
Because $\left(\mathbb{R}^2, < \ldots ; \ldots >\right)$ is a finite dimensional $\mathbb{R}$-vectorspace, there exists by [4] an orthonormal base $\left(e_1, e_2\right) \in \left(\mathbb{R}^2\right)^2$ of $\left(\mathbb{R}^2, < \ldots ; \ldots >\right)$.
Because Φ is an isometry of $\left(\mathbb{R}^2, < \ldots ; \ldots >\right)$, the following is true:

$$\forall u \in \mathbb{R}^2 \quad < u; u > = < \Phi(u); \Phi(u) > \tag{1}$$

With (1) we have:

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ is bijective} \tag{2}$$

and

$$\forall v, w \in \mathbb{R}^2 \quad < v; w > = < \Phi(v); \Phi(w) > \tag{3}$$

Because of [4] and because $(e_1, e_2) \in (\mathbb{R}^2)^2$ is a base of $(\mathbb{R}^2, < \dots ; \dots >)$, there are $a, b, c, d \in \mathbb{R}$ with

$$\Phi(e_1) = ae_1 + ce_2 \quad \text{and} \quad \Phi(e_2) = be_1 + de_2$$

That means with (2)

The matrix $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ is the matrix of the $\mathbb{R}$-linear isometry Φ, which is related to the base (e_1, e_2) of $(\mathbb{R}^2, < \dots ; \dots >)$. (4)

With [4] the following is true:

There is one and only one linear mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ with the property $\forall v, w \in \mathbb{R}^2 \quad < \Phi(v); w > = < v; \varphi(w) >$ (5)

Because $(e_1, e_2) \in (\mathbb{R}^2)^2$ is an orthonormal base, we have with [4] and (5):

The matrix $A^t := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ is the matrix of φ, which is related to the base (e_1, e_2) of $(\mathbb{R}^2, < \dots ; \dots >)$. (6)

On the other hand we have with (2) and (4):

$$0 \neq \det(A) = ad - bc \in \mathbb{R} \tag{7}$$

and

$$\left.\begin{array}{l}\text{The matrix } A^{-1} := \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \\ \text{is the matrix of } \Phi^{-1}\text{, which is related to} \\ \text{the base } (e_1, e_2) \text{ of } (\mathbb{R}^2, < \ldots ; \ldots >).\end{array}\right\} \tag{8}$$

Now we have with (3):

$$\forall v, w \in \mathbb{R}^2 \quad < v; \Phi^{-1}(w) > = < \Phi(v); \Phi\left(\Phi^{-1}(w)\right) >$$

respectively

$$\forall v, w \in \mathbb{R}^2 \quad < v; \Phi^{-1}(w) > = < \Phi(v); w >$$

With (4), (5), (6) and (8) we have:

$$\left.\begin{array}{l}\varphi = \Phi^{-1} \text{ and } A^t = A^{-1} \\ 1 = \det\left(AA^{-1}\right) = \det\left(AA^t\right) = \left(\det(A)\right)^2\end{array}\right\} \tag{9}$$

With that and (7) we have at last:

$$\det(A) \in \{1, -1\} \tag{10}$$

1. case: $\det(A) = -1$

With (6), (8) und (9) we have:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = A^t = A^{-1} = (-1)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

respectively

$$a = -d \text{ und } b = c \tag{*}$$

Let $I_2 \in \mathrm{GL}_2(\mathbb{R})$ be the identity matrix. Now we define the characteristic polynom $\chi_A(\lambda) \in \mathbb{R}[\lambda]$ of A through

$$\chi_A(\lambda) := \det\left(A - \lambda I_2\right)$$

With (4) we have:

$$\chi_A(\lambda) = \det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

respectively

$$\chi_A(\lambda) = ad - bc - (a + d)\lambda + \lambda^2$$

With $\det(A) = -1$, (7) and (*) we have:

$$\chi_A(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

Then we define $\lambda_1, \lambda_2 \in \mathbb{R}$ durch $\lambda_1 := 1$ und $\lambda_2 := -1$. The the following is true:

$$\forall i \in \{1, 2\} \quad \chi_A\left(\lambda_i\right) = 0$$

d.h. λ_1, λ_2 are eigenvalues of A

According to [4] there exists $v_1, v_2 \in \mathbb{R}^2$ with:

$$\forall i \in \{1, 2\} \quad v_i \neq 0$$
$$\forall i \in \{1, 2\} \quad < v_i ; v_i > =1$$
and
$$\forall i \in \{1, 2\} \quad Av_i = \lambda_i v_i$$
d.h. v_1, v_2 are eigenvectors of A

With that we have:

$$\forall i \in \{1, 2\} \quad \Phi\left(v_i\right) = \lambda_i v_i$$

With (3) we have:

$$< v_1 ; v_2 > \; = \; < \Phi\left(v_1\right) ; \Phi\left(v_2\right) >$$

With that we get:

$$< v_1 ; v_2 > \;=\; \lambda_1\lambda_2 < v_1 ; v_2 >$$

Because $\lambda_1\lambda_2 = -1$, it is shown:

$$< v_1 ; v_2 > = 0$$

Then $\left(v_1, v_2\right)$ is an orthonormal base of $\left(\mathbb{R}^2, < \ldots ; \ldots >\right)$. Then we have:

> The matrix $\widetilde{A} := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in \mathrm{GL}_2\left(\mathbb{R}\right)$ is the matrix of the $\mathbb{R}$-linear isometry Φ which is related to $\left(v_1, v_2\right)$.

2. case: $\det(A) = 1$

With (6), (8) and (9) we have:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = A^t = A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

respectively

$$a = d \text{ and } b = -c \qquad (**)$$

Because $\det(A) = 1$ and (7), we have:

$$1 = ad - bc = a^2 + c^2$$

According to [4] there exists $\vartheta \in [0, 2\pi[$ with

$$a = \cos(\vartheta) \text{ and } c = \sin(\vartheta)$$

With (4) and (**), we have at last:

$$A = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$$

„⇐“: Let $(e_1, e_2) \in (\mathbb{R}^2)^2$ be an orthonormal base of $(\mathbb{R}^2, < \ldots ; \ldots >)$. Let $A \in M_{2\times 2}(\mathbb{R})$ with

a) $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

or

b) $\exists \vartheta \in [0, 2\pi[\quad A = \begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$

Let A the matrix of the $\mathbb{R}$-linear mapping $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$, which is related to (e_1, e_2). Then we have:

$$A \in GL_2(\mathbb{R}) \text{ and } A^t = A^{-1} \tag{1}$$

Because (e_1, e_2) is an orthonormal base of $(\mathbb{R}^2, < \ldots ; \ldots >)$, we have:

For the $\mathbb{R}$-linear mapping $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is A^t the matrix of φ, which is related (e_1, e_2), and the following is true: $\forall v, w \in \mathbb{R}^2 \quad < \Phi(v); w > \; = \; < v; \varphi(w) >$ (2)

With (1) and (2) we have:

$$\forall v, w \in \mathbb{R}^2 \quad < \Phi(v); w > \; = \; < v; \Phi^{-1}(w) >$$

respectively

$$\forall v \in \mathbb{R}^2 \quad < \Phi(v); \Phi(v) > \; = \; < v; \Phi^{-1}(\Phi(v)) >$$

respectively

$$\forall v \in \mathbb{R}^2 \quad < \Phi(v); \Phi(v) > \; = \; < v; v >$$

With this it is shown:

Φ is an isometry of $(\mathbb{R}^2, < \ldots ; \ldots >)$

Lemma: **II**

Pre.: Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{R}$-linear mapping.

Ass.:

$$\begin{pmatrix}\text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\mathbb{R}^2, \|\ldots\|\right) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{pmatrix} \Rightarrow$$

$$\begin{pmatrix}\text{There is no scalar product } <\ldots;\ldots> \text{ on } \mathbb{R}^2 \\ \text{and no orthonormal base } \mathfrak{B} \text{ of } \left(\mathbb{R}^2, <\ldots;\ldots>\right), \\ \text{that the matrix of } \Phi, \text{ which is related to } \mathfrak{B}, \\ \text{has the following shape:} \\ \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\end{pmatrix}$$

Proof: We assume now:

$$\begin{pmatrix}\text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\mathbb{R}^2, \|\ldots\|\right) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{pmatrix} \quad (1)$$

Let $<\ldots;\ldots>$ be a scalar product on $\mathbb{R}^2$ and let $\left(e_1, e_2\right) \in \left(\mathbb{R}^2\right)^2$ be an orthonormal base of $\left(\mathbb{R}^2, <\ldots;\ldots>\right)$ und let

$$\Phi\left(e_1\right) = e_1 \text{ and } \Phi\left(e_2\right) = -e_2 \quad (2)$$

Then we define a base $\left(v_1, v_2\right) \in \left(\mathbb{R}^2\right)^2$ von $\left(\mathbb{R}^2, <\ldots;\ldots>\right)$ through

$$v_1 = \frac{1}{\sqrt{2}}\left(e_1 + e_2\right) \text{ and } v_2 = \frac{1}{\sqrt{2}}\left(e_1 - e_2\right) \quad (3)$$

Then we have:

$$\left.\begin{array}{l}\left(v_1, v_2\right) \in \left(\mathbb{R}^2\right)^2 \text{ is an orthonormal base} \\ \text{of } \left(\mathbb{R}^2, < \dots ; \dots >\right)\end{array}\right\} \quad (4)$$

and

$$\Phi\left(v_1\right) = v_2 \text{ und } \Phi\left(v_2\right) = v_1 \quad (5)$$

Now we define a norm $\|\dots\|_1$ on $\mathbb{R}^2$ through

$$\forall u \in \mathbb{R}^2 \quad \|u\|_1 := \frac{1}{2}\left|< u; v_1 >\right| + 2\left|< u; v_2 >\right| \quad (6)$$

With this we have at last:

$$\left\|v_1\right\|_1 = \frac{1}{2} \quad \text{und} \quad \left\|v_2\right\|_1 = 2 \quad (7)$$

(1), (5) are (7) are contradictory!

Lemma: **III**

Pre.: Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{R}$-linear mapping.
Let $< \dots ; \dots >$ be a scalar product on $\mathbb{R}^2$ and let $\left(e_1, e_2\right) \in \left(\mathbb{R}^2\right)^2$ be an orthonormal base of $\left(\mathbb{R}^2, < \dots ; \dots >\right)$.
Let $\vartheta \in [0, 2\pi[$ and let $\begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$ the matrix of Φ which is related to $\left(e_1, e_2\right)$.

Ass.: $\left(\begin{array}{l}\text{For every norm } \|\dots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : \left(\mathbb{R}^2, \|\dots\|\right) \to \left(\mathbb{R}^2, \|\dots\|\right) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \Rightarrow$

$\vartheta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$

Proof: We assume now:

$$\left(\begin{array}{l}\text{For every norm } \|\dots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : \left(\mathbb{R}^2, \|\dots\|\right) \to \left(\mathbb{R}^2, \|\dots\|\right) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \tag{1}$$

Then we define a norm $\|\dots\|_1$ on $\mathbb{R}^2$ through

$$\forall u \in \mathbb{R}^2 \quad \|u\|_1 := \max\left\{\left|< u; e_1 >\right|, \left|< u; e_2 >\right|\right\} \tag{2}$$

Then we have:

$$\left\|e_1\right\|_1 = 1 \quad \text{und} \quad \left\|e_2\right\|_1 = 1 \tag{3}$$

On the other hand the following is true:

$$\left\|\Phi\left(e_1\right)\right\|_1 = \left\|\Phi\left(e_2\right)\right\|_1 = \max\left\{\left|\cos(\vartheta)\right|, \left|\sin(\vartheta)\right|\right\} \tag{4}$$

With (1), (3) and (4) we have:

$$1 = \max\left\{\left|\cos(\vartheta)\right|, \left|\sin(\vartheta)\right|\right\} \tag{5}$$

Because $\vartheta \in [0, 2\pi[$, we get at last:

$$\vartheta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\} \tag{6}$$

Lemma: **IV**

Pre.: Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{R}$-linear mapping.
Let $< \ldots ; \ldots >$ be a scalar product on $\mathbb{R}^2$ and let $(e_1, e_2) \in (\mathbb{R}^2)^2$ be an orthonormal base of $(\mathbb{R}^2, < \ldots ; \ldots >)$.
Let $\vartheta \in [0, 2\pi[$ and let $\begin{pmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{pmatrix}$ the matrix of Φ which is related to (e_1, e_2).

Ass.: $\left(\begin{array}{l} \text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : (\mathbb{R}^2, \|\ldots\|) \to (\mathbb{R}^2, \|\ldots\|) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces} \end{array} \right) \Rightarrow$

$\vartheta \in \{0, \pi\}$

Proof: We assume now:

$$\left(\begin{array}{l} \text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : (\mathbb{R}^2, \|\ldots\|) \to (\mathbb{R}^2, \|\ldots\|) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces} \end{array} \right) \quad (1)$$

According to Lemma III we have:

$$\vartheta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\} \quad (2)$$

Then we have to proof:

$$\vartheta \in \{0, \pi\} \quad (3)$$

Ass.: $\vartheta = \frac{\pi}{2}$ or $\vartheta = \frac{3\pi}{2}$ (4)

Then there exists $\alpha \in \{1, -1\}$, that $\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of Φ which is related to (e_1, e_2).
Then we have:

$$\Phi(e_1) = \alpha e_2 \text{ and } \Phi(e_2) = -\alpha e_1 \qquad (5)$$

We now define a norm $\|\ldots\|_1$ on $\mathbb{R}^2$ through

$$\forall u \in \mathbb{R}^2 \quad \|u\|_1 := \frac{1}{2}\left|< u; e_1 >\right| + 2\left|< u; e_2 >\right| \qquad (6)$$

For this norm the following is true:

$$\|e_1\|_1 = \frac{1}{2} \text{ and } \|e_2\|_1 = 2 \qquad (7)$$

(1), (5) and (7) are contradictory!

Lemma: V (Consequence of Lemma I, II, III and IV)

Pre.: Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathbb{R}$-linear mapping.

Ass.:

$$\left(\begin{array}{l}\text{For every norm } \|...\| \text{ on } \mathbb{R}^2 \text{ is true:} \\ \Phi : \left(\mathbb{R}^2, \|...\|\right) \to \left(\mathbb{R}^2, \|...\|\right) \text{ is an isometry} \\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \Leftrightarrow$$

$$\Phi \in \left\{\mathrm{id}_{\mathbb{R}^2}, -\mathrm{id}_{\mathbb{R}^2}\right\}$$

Proof of 4.3.

Obviously we have to proof:

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:}\\ \Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{**}, \|\ldots\|_{**}\right) \text{ is an isometry}\\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \Rightarrow$$

$$\Phi \in \left\{Q_{\mathbb{R}^2}, -Q_{\mathbb{R}^2}\right\}$$

Proof:

We assume:

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:}\\ \Phi : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\left(\mathbb{R}^2\right)^{**}, \|\ldots\|_{**}\right) \text{ is an isometry}\\ \text{of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \qquad (1)$$

With this we have:

$$\left(\begin{array}{l}\text{For every norm } \|\ldots\| \text{ on } \mathbb{R}^2 \text{ is true:}\\ \left(\left(Q_{\mathbb{R}^2}\right)^{-1} \circ \Phi\right) : \left(\mathbb{R}^2, \|\ldots\|\right) \to \left(\mathbb{R}^2, \|\ldots\|\right) \text{ is an}\\ \text{isometry of normed } \mathbb{R}\text{-vectorspaces}\end{array}\right) \qquad (2)$$

This means with Lemma V:

$$\left(\left(Q_{\mathbb{R}^2}\right)^{-1} \circ \Phi\right) \in \left\{\mathrm{id}_{\mathbb{R}^2}, -\mathrm{id}_{\mathbb{R}^2}\right\} \qquad (3)$$

Then there exists $a \in \{1, -1\}$ with

$$\left(Q_{\mathbb{R}^2}\right)^{-1} \circ \Phi = a \cdot \mathrm{id}_{\mathbb{R}^2} \tag{4}$$

Because $Q_{\mathbb{R}^2}$ is a $\mathbb{R}$-linear mapping, we get:

$$\underbrace{Q_{\mathbb{R}^2} \circ \left(Q_{\mathbb{R}^2}\right)^{-1}}_{=\mathrm{id}_{\left(\mathbb{R}^2\right)^{**}}} \circ \Phi = a \cdot Q_{\mathbb{R}^2} \tag{5}$$

Because $a \in \{1, -1\}$, we have at last:

$$\Phi \in \left\{Q_{\mathbb{R}^2}, -Q_{\mathbb{R}^2}\right\} \tag{6}$$