1. Symmetry of the 2^{nd} Differential

Theorem:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^n$.
Let $\varphi : G \to \mathbb{R}$ be a mapping.
Let φ be twice differentiable.

Ass.: $\forall p \in G \quad d_p^2\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric

2. A Special Case of Cartan's Derivation

Theorem:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^n$.

Let $V : G \to \mathbb{R}^n$ be a continuous differentiable mapping.

Let $\omega : G \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ be defined as $\omega := \sum_{i=1}^{n} V_i \cdot dx_i$

(espacially $\omega : G \to \mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ is continuous differentiable).

Ass.: $\forall p \in G \quad \left(\mathfrak{d}_p\omega = 0 \quad \Leftrightarrow \quad d_pV \text{ ist self-adjoint}\right)$

Rem.:

1. $\mathfrak{L}\left(\mathbb{R}^n, \mathbb{R}\right)$ is the $\mathbb{R}$-vector-space of all $\mathbb{R}$-linear-forms $\mathbb{R}^n \to \mathbb{R}$.

2. ω is a so called C^1-differential form of degree 1.

3. $\mathfrak{d}\ldots$ is Cartan's derivation. If $n = 3$, then the following is true:
   $\forall p \in G \quad \left(\left(\mathfrak{d}_p\omega = 0\right) \quad \Leftrightarrow \quad \left(\text{curl}_p\left(V\right) = 0\right)\right).$

3. Vector Potential

Theo.:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open subset of $\mathbb{R}^n$.

Ass.:

1. Let $\varphi \in C^2(G)$.
   Then the follwing is true:
   $\left(\forall p \in G \quad \left(d_p\left(\text{grad}(\varphi)\right) \text{ ist self-adjoint}\right)\right)$

2. Be G star-shaped.
   Be $k \in \mathbb{N}_+ \cup \{\infty\}$.
   Let $V: \ G \to \mathbb{R}^n$ be a k-times continuous differentiable mapping.
   Then the following is true:
   $\left(\forall p \in G \quad \left(d_p V \text{ ist self-adjoint}\right)\right) \quad \Rightarrow$
   $\exists \varphi \in C^{k+1}(G) \quad V = \text{grad}(\varphi)$

4. The Inversal of 1

Theorem:

Pre.: Let $n \in \mathbb{N}_+$.

Let G be an open, star-shaped subset of $\mathbb{R}^n$.

Let $\alpha : G \to \mathfrak{L}(\mathbb{R}^n, \mathbb{R})$ be a continuous differentiable mapping.

Let $\beta : G \to \mathfrak{L}^2(\mathbb{R}^n, \mathbb{R})$ be defined through

$\forall p \in g \quad \forall v, w \in \mathbb{R}^n \quad (\beta_p)(v, w) := ((\mathrm{d}_p \alpha)(v))(w)$

Ass.: $\left(\forall p \in G \quad (\beta_p \text{ ist symmetric})\right) \quad \Rightarrow$

$\begin{pmatrix} \text{There is } \varphi \in C^2(G) \text{ with} \\ \mathrm{d}\varphi = \alpha \text{ and } \mathrm{d}^2\varphi = \beta \end{pmatrix}$

Rem.: $\mathfrak{L}^2(\mathbb{R}^n, \mathbb{R}) = \{b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ is bilinear}\}$ is a $\mathbb{R}$-vector-space.

Proof: Let $\mathfrak{E} := (e_1, \dots, e_n)$ be the standard base of $\mathbb{R}^n$.

Let $\mathfrak{X} := (x_1, \dots, x_n)$ be the base of $\mathfrak{L}(\mathbb{R}^n, \mathbb{R})$ dual to $\mathfrak{E}$.

Let the mapping $V : G \to \mathbb{R}^n$ be defined as

$$\forall p \in G \quad V(p) := \begin{pmatrix} \alpha_p(e_1) \\ \vdots \\ \alpha_p(e_n) \end{pmatrix} \tag{1}$$

Then the following statements are valid:

$$V : G \to \mathbb{R}^n \text{ is continuous differentiable} \tag{2}$$

and

$$\forall p \in G \quad \mathrm{d}_p V = \begin{pmatrix} \mathrm{d}_p(\alpha_{\dots}(e_1)) \\ \vdots \\ \mathrm{d}_p(\alpha_{\dots}(e_n)) \end{pmatrix} = \begin{pmatrix} (\mathrm{d}_p \alpha)(e_1) \\ \vdots \\ (\mathrm{d}_p \alpha)(e_n) \end{pmatrix} \tag{3}$$

Then the following statement is true:

$$\forall j \in \{1, \ldots, n\} \quad \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \frac{\partial V}{\partial x_j} \tag{4}$$

A consequence of (3) is

$$\forall j \in \{1, \ldots, n\} \quad \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \begin{pmatrix} \left(\mathrm{d}_p \alpha\right)\left(e_1\right) \\ \vdots \\ \left(\mathrm{d}_p \alpha\right)\left(e_n\right) \end{pmatrix} \left(e_j\right)$$

Then it follows by premise:

$$\forall j \in \{1, \ldots, n\} \quad \forall p \in G \quad \left(\mathrm{d}_p V\right)\left(e_j\right) = \begin{pmatrix} \left(\beta_p\right)\left(e_1, e_j\right) \\ \vdots \\ \left(\beta_p\right)\left(e_n, e_j\right) \end{pmatrix} \tag{5}$$

Because $\left(\forall p \in G \ \left(\beta_p \text{ ist symmetric}\right)\right)$, a consequence of (4) and (5) is

$$\left(\forall p \in G \ \left(\mathrm{d}_p V \text{ ist self-adjoint}\right)\right) \tag{6}$$

Because of (2), (6) and theroem 3.2., there exists $\varphi \in \mathrm{C}^2(G)$ with

$$V = \mathrm{grad}(\varphi) \tag{7}$$

Because of (1) and (7), for all $\forall i \in \{1, \ldots, n\}$ and all $\forall p \in G$ the following statement is valid:

$$\left(\mathrm{d}_p \varphi\right)\left(e_i\right) = \frac{\partial \varphi}{\partial x_i}(p) = \left(\mathrm{grad}_p(\varphi)\right)_i = V_i(p) = \alpha_p\left(e_i\right)$$

respectively

$$\mathrm{d}\varphi = \alpha \tag{8}$$

Then it follows for all $\forall i, j \in \{1, \ldots, n\}$ and all $\forall p \in G$

$$\mathrm{d}_p^2\varphi\left(e_i, e_j\right) = \left(\mathrm{d}_p\left(\left(\mathrm{d}_{\ldots}\varphi\right)\left(e_i\right)\right)\right)\left(e_j\right) = \left(\mathrm{d}_p\left(\left(\alpha_{\ldots}\right)\left(e_i\right)\right)\right)\left(e_j\right)$$

respectively

$$\mathrm{d}_p^2\varphi\left(e_i, e_j\right) = \left(\left(\mathrm{d}_p(\alpha)\right)\left(e_i\right)\right)\left(e_j\right)$$

respectively by premise

$$\mathrm{d}_p^2\varphi\left(e_i, e_j\right) = \left(\beta_p\left(e_i, e_j\right)\right)$$

At last we have

$$\mathrm{d}^2\varphi = \beta \tag{9}$$