1. Excerpts from the Axiomatic Set Theory (ZFC) (generalized)

1.1. General Premises

We discuss a domain $\mathfrak{M}$ of objects, which we call „sets". In the future context every letter A, B, M, X, Y and Z symbolize a set.

Let α be a two-place statement form on the domain $\mathfrak{M}$, i. e. for every two sets X, Y it is certain, wether $X\alpha Y$ is valid or $X\alpha Y$ is not valid (i. e. $X\not\alpha Y$ is valid). Further we assume

$$\left(\forall A,B \quad (A\alpha B \quad \Leftrightarrow \quad A\in B)\right) \quad \vee \quad \left(\forall A,B \quad (A\alpha B \quad \Leftrightarrow \quad A\notin B)\right)$$

Further let = be a tow-place statment form on the domain $\mathfrak{M}$, i. e. for every two sets X, Y it is certain, wether $X=Y$ is valid or $X=Y$ is not valid (i. e. $X\neq Y$ is valid). In addition to this = shall have the following properties:

1. $\forall X \quad X=X$
2. $\forall X,Y \quad (X=Y \quad \Rightarrow \quad Y=X)$
3. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad Y=Z) \quad \Rightarrow \quad X=Z)$
4. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad X\alpha Z) \quad \Rightarrow \quad Y\alpha Z)$

A consequence of 1. - 4. is especially:

5. $\forall X,Y,Z \quad ((X=Y \quad \wedge \quad X\not\alpha Z) \quad \Rightarrow \quad Y\not\alpha Z)$

1.2. Axiom of Existence

Axiom:

Ass.: There exists a set M with the property

$$\forall X \quad X \not\alpha M$$

1.3. Axiom of Extension

Axiom:

Ass.: For all sets A, B the following stament is valid:

$$\left(\forall X \quad \left(X \alpha A \quad \Leftrightarrow \quad X \alpha B\right)\right) \quad \Rightarrow \quad A = B$$

Rem.:

1. With 1.1. it is possible to proof:

   $$\left(\forall X \quad \left(X \alpha A \quad \Leftrightarrow \quad X \alpha B\right)\right) \quad \Leftrightarrow \quad A = B$$

2. The following statment is true:

   $$\left(\forall X \quad \left(X \alpha A \quad \Leftrightarrow \quad X \alpha B\right)\right) \quad \Leftrightarrow \quad \left(\forall X \quad \left(X \not\alpha A \quad \Leftrightarrow \quad X \not\alpha B\right)\right)$$

3. With the axiom of existence and the axiom of extension it is possible to proof:

   There exist one and only one set M with the property

   $$\forall X \quad X \not\alpha M$$

1.4. Axiom-Scheme of Comprehension (out of date)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statment form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, wether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: There exists a set B with the property:

$$\forall X \quad (X \alpha B \quad \Leftrightarrow \quad P(X))$$

Rem.: 1. With the axiom schema of comprehension and the axiom of extension it is possible to proof:

There exists one and only one set B with the property:

$$\forall X \quad (X \alpha B \quad \Leftrightarrow \quad P(X))$$

For this certain B we write $\{P(X)\}_\alpha$.

2. "Russel's Antinomy":

With this Axiom $\{X \not\alpha X\}_\alpha$ would be a set. With Russel this leads to a contradiction.

1.5. Axiom-Scheme of Comprehension (present)

Axiom:

Pre.: Let $P(\ldots)$ be an one-place statment form on the domain $\mathfrak{M}$, i. e. for every set X it is certain, wether $P(X)$ is valid or $P(X)$ is not valid (i. e. $\neg(P(X))$ is valid).

Ass.: For every set A there exists a set B with the property:

$$\forall X \ \left(X \alpha B \ \Leftrightarrow \ \left(X \alpha A \ \wedge \ P(X) \right) \right)$$

Rem.: With the axiom schema of comprehension and the axiom of extension it is possible to proof:

For every set A there exists one and only one set B with the property:

$$\forall X \ \left(X \alpha B \ \Leftrightarrow \ \left(X \alpha A \ \wedge \ P(X) \right) \right)$$

For this certain B we write $\{X \alpha A: \ P(X)\}_{\alpha}$.

1.6. Theorem

Theorem:

Ass.: There does not exist a set M with the property

$\forall X \quad X \alpha M$

Proof:

Supp.: There exists a set M with the property

$$\forall X \quad X \alpha M \tag{1}$$

Because M is a set, with the Axiom-Schema of comprehension we have (Cave! $P(X) := X \not\alpha X$ defines a one-place statment form (see general premises)):

$$A := \{X \alpha M : \; X \not\alpha X\}_{\alpha} \text{ is a set} \tag{2}$$

Now we have by (1) with (2):

$$A \alpha M \tag{3}$$

Finally the following statement is valid:

$$A \alpha A \text{ or } A \not\alpha A \tag{4}$$

1st case: $A \alpha A$ is true.

Then with the definition of A we have:

$$A \not\alpha A$$

This is a contradiction!

2nd case: $A \not\alpha A$ is true.

Then with (3) and the definition of A we have:

$$A \alpha A$$

This is a contradiction!

2. Result

If the α in chapter 1 has the property $\forall A,B\ (A\alpha B \ \Leftrightarrow\ A \in B)$, you get:

1. There exists a set M with the property

   $\forall X \quad X \notin M$

2. There does not exist a set M with the property

   $\forall X \quad X \in M$

If the α in chapter 1 has the property $\forall A,B\ (A\alpha B \ \Leftrightarrow\ A \notin B)$, you get:

3. There exists a set M with the property

   $\forall X \quad X \in M$

4. There does not exist a set M with the property

   $\forall X \quad X \notin M$

The 3 axioms of chapter 1 are incompatible.

3. Further Thoughts

3.1. Alternative Set Theory

If one looks at 1. and 2., it comes to his mind, that one gets another set theory $\notin$ −ZFC (which is different to ZFC), if one replaces all occurrencies of $\in$ with $\notin$ and all occurrencies of $\notin$ with $\in$ in the axioms of ZFC. This set theory $\notin$ −ZFC is dual to ZFC and at the same time with it free of contradiction or not free of contradiction.

In the standard set theory ZFC one gets a set M, if he begins with the empty set and adds all the elements of M. Exactly it means:

$$\forall n \in \mathbb{N}_+ \ \ \forall A_1,\ldots,A_n \ \ \forall X \ \left(X \in \left\{ A_1,\ldots,A_n \right\}_{\in} \ \ \Leftrightarrow \ \ \left(X = A_1 \ \vee \ \ldots \ \vee \ X = A_n \right) \right)$$

In the alternative set theory $\notin$ −ZFC one gets a set M, if he begins with the set of all sets and subtracts all the non-elements of M. Exactly it means:

$$\forall n \in \mathbb{N}_+ \ \ \forall A_1,\ldots,A_n \ \ \forall X \ \left(X \notin \left\{ A_1,\ldots,A_n \right\}_{\notin} \ \ \Leftrightarrow \ \ \left(X = A_1 \ \vee \ \ldots \ \vee \ X = A_n \right) \right)$$

respectively

$$\forall n \in \mathbb{N}_+ \ \ \forall A_1,\ldots,A_n \ \ \forall X \ \left(X \in \left\{ A_1,\ldots,A_n \right\}_{\notin} \ \ \Leftrightarrow \ \ \left(X \neq A_1 \ \wedge \ \ldots \ \wedge \ X \neq A_n \right) \right)$$

3.2. Set „Monster"

The thoughts of 3.1. lead to the result, that the empty set is a „Monster" at the same time with the set of all set.

3.3. ZFC is not "canonical"

Because the axioms of ZFC (written with α like in 1.) only use $\alpha = \neg\not\alpha$ and $\not\alpha = \neg\alpha$, ZFC and $\notin$−ZFC at the same time both are free of contradiction or both are not free of contradiction. If one assumes ZFC, ZFC (and of course $\notin$−ZFC) is **not "canonical"**.

3.4. ZFC is not valid

If one assumes, that ZFC is valid, $\notin$−ZFC is valid too. Then one gets (look at 1. and 2.), that the first three axioms of ZFC are incompatible.

3.5. Totally Alien Language

If the chapter 1. was written in a totally alien language, one could not distinguish between the in chapter 1. defined two variants ZFC or $\notin$−ZFC.

4. Index of Literature

[1] Lectures in Mathematics 1987 – 2002
University of Cologne (Germany)