1. Tools

Def.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0; 1] \quad \phi(tx + (1 - t)y) \le t\phi(x) + (1 - t)\phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex

Rem.: Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotonically increasing$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \ge 0$

2. Gamma-Function

The Gamma-Funktion $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ is for all $\alpha \in \mathbb{R}_+$ defined through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is analytically} \quad (1)$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha + 1) = \alpha \cdot \Gamma(\alpha) \quad (2)$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k + 1) = k! \quad (3)$$

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \quad (4)$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \quad (5)$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[\text{ is monotonically increasing} \quad (6)$$

We now define a mapping $\gamma :]-1; \infty[\rightarrow \mathbb{R}$ through

$$\forall u \in]-1; \infty[\quad \gamma(u) := \Gamma(u + 1)$$

Then we have with (2):

$$\forall v \in]-1; \infty[\quad \gamma(v + 1) = (v + 1)\,\gamma(v) \quad (7)$$

In addition we have with (6):

$$\gamma \mid [1; \infty[\text{ is monotonically increasing} \quad (8)$$

3. A Look at the ln-Function

Let $J :=]0;1[$
We define a function $f : J \to \mathbb{R}$ through

$$\forall t \in J \quad f(t) := \ln(1-t)$$

The following is known:

$$f : J \to \mathbb{R} \text{ is differentiable}$$

$$\forall t \in J \quad f'(t) = -\frac{1}{1-t} = -\sum_{n=0}^{\infty} t^n$$

$$\forall t \in J \quad f(t) = -\sum_{n=0}^{\infty} \frac{1}{n+1} t^{n+1} = -\sum_{n=0}^{\infty} \frac{n!}{(n+1)!} t^{n+1}$$

Now we define for $\tilde{\alpha} \in]-1;\infty[$ the function $l_{\tilde{\alpha}} : J \to \mathbb{R}$ through

$$\forall t \in J \quad l_{\tilde{\alpha}}(t) := -\sum_{n=0}^{\infty} \frac{\gamma(n+\tilde{\alpha})}{\gamma(n+\tilde{\alpha}+1)} t^{n+\tilde{\alpha}+1}$$

Then we have for all $\tilde{\alpha} \in]-1;\infty[$:

$$l_{\tilde{\alpha}} : J \to \mathbb{R} \text{ is well-defined and differentiable}$$

and

$$\forall t \in J \quad \left(l_{\tilde{\alpha}}\right)'(t) = -\sum_{n=0}^{\infty} \frac{\gamma(n+\tilde{\alpha})}{\gamma(n+\tilde{\alpha}+1)} \left(x^{n+\tilde{\alpha}+1}\right)'(t) =$$

$$= -\sum_{n=0}^{\infty} t^{n+\tilde{\alpha}} =$$

$$= \left(-\sum_{n=0}^{\infty} t^n\right) t^{\tilde{\alpha}} =$$

$$= \frac{t^{\tilde{\alpha}}}{t-1}$$

4. Literature

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